NEW YORK UNIVERSITY
WASHINGTON SQUARE COLLEGE
MATHEMATICS RESEARCH GROUP

RESEARCH REPORT NO. 172-3

# The Propagation of Electromagnetic Plane Waves in Plane Parallel Layers

by

R. K. LUNEBERG

This Report was Prepared and Written Under
Contract No. W 28-099-ac-172
with
Army Air Forces, Air Materiel Command
Watson Laboratories
Red Bank, N. J.

JUNE 1947

New York University
Washington Square College
Mathematics Research Group

# THE PROPAGATION OF ELECTROMAGNETIC PLANE WAVES IN PLANE PARALLEL LAYERS

By
R. K. Luneberg
Luneburg

Contract No. W28-099-ac-172
Research Report 172-3

Submitted by
Hollis R. Cooley
Hollis R. Cooley
Technical Editor

Title page
50 numbered pages
30 June, 1947

CONTENTS


| Article | Page |
|---|---|
| Abstract | 2 |
| 1. Introduction. | 3 |
| 2. The Differential Equations. | 6 |
| 3. Continuity Conditions. | 10 |
| 4. The Multilayer. | 11 |
| 5. Interpretation as Refracted Plane Waves. | 18 |
| 6. Continuous Variation of $\varepsilon$, $\mu$, and $\sigma$. | 19 |
| 7. The Electromagnetic Field in the Continuous Layer. | 25 |
| 8. Reflection and Transmission of the Continuous Layer. | 30 |
| 9. Reflection and Transmission of the Multilayer. | 32 |
| 10. The Case of Normal Incidence. | 35 |
| 11. Conservation of Energy. | 38 |
| 12. Two Media Separated by One Surface. Fresnel's Formulae. | 46 |
| 13. Three Media Separated by Two Plane Surface. | 47 |
| 14. Conclusion. | 49 |

## Abstract

The following problem is first considered: A plane wave originating in a semi-infinite layer with one plane bounding surface, with constant values of the dielectric constant $\varepsilon$, the magnetic permeability $\mu$, and the conductivity $\sigma$, enters a series of r parallel plane layers in each of which $\varepsilon, \mu, \sigma$ are constant but vary from layer to layer. Expressions are found giving the field at any point. Recursion formulas are derived for the amplitudes of transmitted and reflected waves, including that of the wave transmitted through the whole multilayer system and that of the wave reflected by the whole system.

We next consider the situation in which the multilayer is replaced by a single layer in which $\varepsilon, \mu$, and $\sigma$ may vary continuously as functions of one variable - the distance perpendicular to the bounding planes. This problem is approached by considering the given layer as a limiting case of a multilayer with the thickness of each of its constituent layers approaching 0 as their number becomes infinitely great. The field expressions are thus envelopes of the infinity of multilayer expressions. The solutions are obtained from differential equations of Riccati type instead of recursion formulas. Reflection and transmission coefficients are found. It is also shown that the results for the multilayer can be obtained as special cases from the results for the layer with varying parameters. The calculation of the intensities of reflected and transmitted waves is simplified for many practical cases by formulas derived from the statement of conservation of energy. Important special cases are considered, in particular that of normal incidence.

## 1. Introduction

The present paper represents a first step in a new attack on the problem of propagation of electromagnetic waves in non-homogeneous atmospheres. The general attack contemplates the integration of effects due to plane waves which will lead to a complex integral to be evaluated by one or more methods, for example, that of steepest descent. This attack is opposed to the reduction of the problem to that of solving a partial differential equation with boundary conditions leading to the determination of eigenvalues. It should be possible in this generalization not only to consider the behavior of radiation from a dipole in a non-homogeneous medium but also the radiation of an arbitrary source.

The first step consists of the study of the behavior of plane waves in media with continuously varying electromagnetic parameters, namely the dielectric constant, permeability, and the conductivity. Specifically, this paper will study the behavior of a plane wave which enters a layer in which these parameters vary continuously, the layer to be bounded by parallel planes. In order to approach this problem a preliminary problem is first discussed thoroughly, namely the propagation of a plane wave incident upon a series of parallel plane layers, i.e.,layers lying one upon the other, each bounded by planes, in each of which the electromagnetic parameters are constant. This is the problem of the multilayer. From this study it is possible to pass to the case of a continuously varying layer by allowing the number of parallel plane layers to become infinite while each becomes infinitely thin. It is assumed in this theory, as well as in the generalization referred to, that the electromagnetic parameters vary in one dimension only, namely that which is perpendicular to the layers. It should be remarked, too, that, while the case of a continuously varying medium is generally regarded as the ultimate goal in the study of radio wave propagation under actual meteorological conditions, the study of a finite number of layers, each of definite but arbitrary thickness, may be of equal importance in the practical propagation problem, for it may be desirable from a computational standpoint to approximate a continuous distribution by a finite number of layers in each of which the electromagnetic parameters are constant.

The present paper represents more than a first step in the direction of the general theory of propagation in non-homogeneous media.

Both the preliminary problem of the paper, namely the study of the passage of a plane wave through a multilayer, and the main problem of the propagation of a plane wave through a layer of continuously varying electromagnetic parameters, are generalizations of that portion of standard electromagnetic theory which treats the behavior of a plane wave striking a plane boundary between two semi-infinite media.* As this paper will show, many of the results here can immediately be specialized to results found in Stratton. In particular, one may mention the laws of reflection and refraction in the present paper which reduce to the well known Snell and Fresnel laws cited in Stratton. The present paper is an extension of standard theory also in the sense that the three electromagnetic parameters, i.e., dielectric constant, permeability and conductivity, assume arbitrary values in each layer in the case of the multilayer, and have arbitrary functional form, including finite discontinuities, in the case of the layer of "continuously" varying parameters.

A theory on the propagation of plane waves through layers has application in ultra-high-frequency engineering independently of its use in the theory of propagation through the atmosphere. As an example, often a surface is coated with a layer of some substance so as to absorb incident radiation, which can usually be regarded as being a plane wave. The air, the coating, as well as the surface coated, constitute three layers and the present theory can be applied to determine the behavior of the incident radiation. In particular, the reflection and transmission through the layers and the polarization of the reflected radiation can be determined. The present theory can also be used, though developments of this nature have not actually been carried out in this paper, to study the polarization of reflected waves from a series of layers and to ascertain the conditions under which one may retain the linear polarization of the reflected wave on the assumption that the incident radiation is also linearly polarized.

In the opening part of the paper we consider the behavior of a plane wave incident upon a multilayer. Expressions for the field in each layer are deduced and it is shown how to compute the amplitudes of the electric and magnetic fields reflected by the entire system as well as the amplitudes for the radiation transmitted through the whole multilayer. The treatment of the multilayer also contains extensions of Snell's Laws of Reflection and Refraction.

---

* The more limited standard results may be found in Chapter 9 of Stratton, J.A. Electromagnetic Theory, pages 490 to 524.

From a discussion of a multilayer we proceed to the case of the plane wave incident upon a medium bounded by parallel planes within which the electromagnetic parameters vary continuously or have a finite number of finite discontinuities. Expressions for the electromagnetic field in the layer of "continuously" varying parameters are obtained, as are some properties of the wave in the medium. We obtain formulas for the amplitudes of the electric and magnetic fields at any point in the medium. Formulas for the direction of the wave front at any point in the medium constitute an extension of Snell's Law of Refraction to a medium with continuously varying electromagnetic parameters.

In the theory of the continuous layer, reflection and transmission coefficients, expressing the ratios of the amplitudes of the electric fields reflected and transmitted to that of the incident field, are introduced and computed. More exactly a reflection and a transmission coefficient are introduced for each component of the incident electric field, one parallel to the plane of incidence and the other perpendicular to it. Similar reflection and transmission coefficients are computed for the multilayer. The values of these coefficients for the special but very important case of a plane wave normally incident on the layers is treated in some detail. The paper then shows how, through energy considerations, one may obtain simpler formulas for the transmission and reflection coefficients, in the practical case where the medium carrying the incident wave is a dielectric.

The theory of the multilayer is then applied by way of illustration to two special cases, the first being that of two semi-infinite media separated by a plane, which is the standard case to be found in Stratton, and the case of three media separated by two parallel plane surfaces.

It has been noted that the theory developed here for the layer of continuously varying parameters includes the case where the parameters have a finite number of finite discontinuities. Hence the theory for the "continuous" layer actually includes within itself the theory for the multilayer and all results for the multilayer could have been deduced as a special case of the former theory. The separate approach to the multilayer is given not merely to base treatment of the multilayer on simpler theory but also to provide an intuitive basis for the treatment of the "continuous" case.

While no exhaustive search of the literature has been made in behalf of this particular paper, practically all the results obtained herein are believed to be new. Reference should be made, however, to the paper by M. E. Rose on "The Specular Reflection of Plane Wave Pulses in Media of Continuously Variable Refractive Properties", Physical Review, V. 63, Nos. 3 & 4, Feb. 1943, pp. 111-120. This paper overlaps the present one in that it obtains expressions for a sound wave entering a series of plane layers and obtains limited results on a reflection coefficient for this case and the case of a sound wave entering a layer with continuously varying dielectric constant.

The writer's first work in the theory of this specific paper was actually undertaken several years ago in behalf of some optical problems on which he was then engaged. The theory contained herein can indeed be applied to many useful optical problems such as the effect of coating glass with thin dielectric or metallic films or the effect of joining several types of glass layers. Such applications are now not too far removed from the problems likely to be encountered in microwave engineering, for the use of dielectrics as microwave lenses is already a familiar technique.

## 2. The Differential Equations.

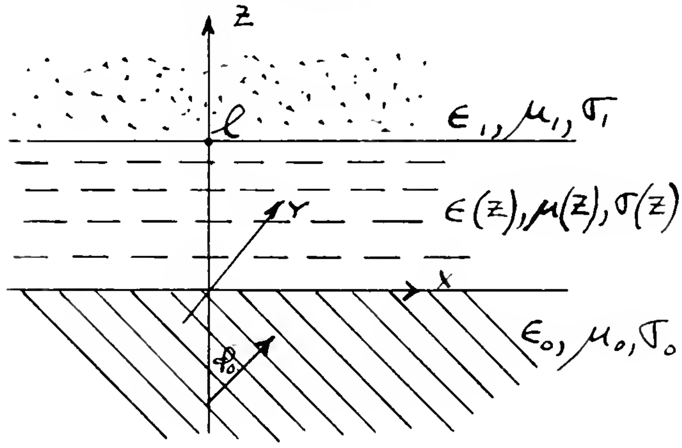



Fig. 1

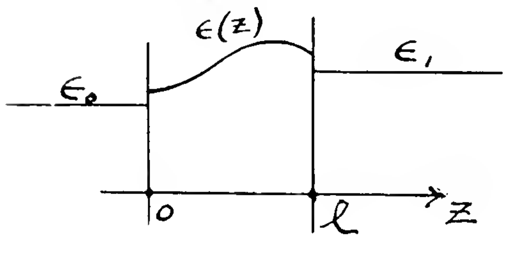



Fig. 2

We assume that the x,y,z-space is occupied by substances which are distributed in infinite layers parallel to the x,y-plane. In order to make the situation more definite we assume homogeneous substances in the halfspaces

$$z < 0 \text{ and } z > \ell .$$

In the remaining space

$$0 < z < \ell$$

the substances are characterized by three functions of z:

Dielectric Constant: $\varepsilon = \varepsilon(z)$.

Magnetic Permeability: $\mu = \mu(z)$.

Electric Conductivity: $\sigma = \sigma(z)$.

No other assumptions about these functions are made but that the interval $0 < z < \ell$ can be divided into a finite number of subintervals in each of which $\varepsilon, \sigma, \mu$ are continuous and have continuous derivatives. The constant values of $\varepsilon, \mu, \sigma$ in the two boundary media are called

$$\varepsilon_0, \mu_0, \sigma_0 \quad \text{and} \quad \varepsilon_1, \mu_1, \sigma_1 \ .$$

We assume that a plane wave approaches our medium from below (Fig. 1). The normal of the wavefronts makes an angle $\varphi_0$ with the z-axis. The problem is to investigate the effect which our layer system has on the undisturbed propagation of the wave.

We base our considerations on Maxwell's Equations

$$\text{(1)} \qquad \begin{aligned} \operatorname{curl} H - \frac{\varepsilon}{c} E_t &= \frac{4\pi\sigma}{c} E, \\ \operatorname{curl} E + \frac{\mu}{c} H_t &= 0, \end{aligned}$$

in which

$$E = (E_1, E_2, E_3)$$

and

$$H = (H_1, H_2, H_3)$$

are the electric and magnetic vectors of the field, respectively. These vectors E and H are, in general, functions of x, y, z and t.

It can be shown that in case of a plane wave radiated by an harmonic oscillator of angular frequency

$$\omega = \frac{2\pi c}{\lambda},$$

periodic equilibrium is established with increasing time t. In effect we are dealing with the steady state, disregarding transient effects. The corresponding vector functions E and H which describe this equilibrium are found to have the form

$$\text{(2)} \qquad \begin{aligned} E &= u\, e^{i\omega t}, \\ H &= v\, e^{i\omega t}, \end{aligned}$$

in which

$$u = (u_1, u_2, u_3),$$

$$v = (v_1, v_2, u_3),$$

are vectors which depend on x, y, z but not on t.

We introduce (2) in Maxwell's Equations (1) and obtain the differential equations:

$$
(3) \qquad \begin{aligned} &\operatorname{curl} v - \frac{i\omega}{c} m^2 u = 0, \\ &\operatorname{curl} u + \frac{i\omega}{c} \mu v = 0. \end{aligned}
$$

The quantity $m = n - ink$, the <u>complex index of refraction</u>, is defined by

$$
(4) \qquad m^2 = \varepsilon - \frac{4\pi\sigma}{\omega} i.
$$

This leads to the relations:

$$
n^2(1 - k^2) = \varepsilon,
$$

$$
kn^2 = \frac{2\pi\sigma}{\omega} = \frac{\sigma\lambda}{c} .
$$

The plane of incidence is, by definition, the plane containing the normal to the boundary surface, i.e. the xy-plane, and the normal to the wave-front. Without loss of generality we may assume that the xz-plane coincides with the plane of incidence of the incident wave. It follows from symmetry considerations that the vectors u and v are independent of y, i.e., <u>are functions of x and z only</u>.

For vector functions of this type the curl-operation simplifies to

$$
\operatorname{curl} u = \left( -\frac{\partial u_2}{\partial z},\ \frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x},\ \frac{\partial u_2}{\partial x} \right),
$$

$$
\operatorname{curl} v = \left( -\frac{\partial v_2}{\partial z},\ \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x},\ \frac{\partial v_2}{\partial x} \right),
$$

and we thus obtain the following set of differential equations from (3):

$$
(5) \qquad \begin{aligned} &-\frac{\partial v_2}{\partial z} = \frac{i\omega}{c} m^2 u_1, && -\frac{\partial u_2}{\partial z} = -\mu \frac{i\omega}{c} v_1, \\ &\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} = \frac{i\omega}{c} m^2 u_2, && \frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x} = -\mu \frac{i\omega}{c} v_2, \\ &\frac{\partial v_2}{\partial x} = \frac{i\omega}{c} m^2 u_3, && \frac{\partial u_2}{\partial x} = -\mu \frac{i\omega}{c} v_3 . \end{aligned}
$$

We conclude from these equations: The components $u_1$, $u_3$; $v_1$, $v_3$ are completely determined by the components $u_2$ and $v_2$ :

$$
(6)\qquad
\begin{aligned}
u_1 &= i\,\frac{c}{m^2\omega}\,\frac{\partial v_2}{\partial z}, \qquad & v_1 &= -\frac{c}{\mu\omega}\,i\,\frac{\partial u_2}{\partial z},\\
u_3 &= -i\,\frac{c}{m^2\omega}\,\frac{\partial v_2}{\partial x}, \qquad & v_3 &= \frac{c}{\mu\omega}\,i\,\frac{\partial u_2}{\partial x}.
\end{aligned}
$$

Using (6) we can eliminate $u_1,u_3$; $v_1,v_3$ in the second row of equations (5). As a result we have two partial differential equations of second order for $u_2$ and $v_2$ :

$$
(7)\qquad
\begin{aligned}
&\frac{\partial}{\partial x}\left(\frac{1}{\mu}\,\frac{\partial u_2}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{1}{\mu}\,\frac{\partial u_2}{\partial z}\right) + \frac{m^2\omega^2}{c^2}\,u_2 = 0,\\
&\frac{\partial}{\partial x}\left(\frac{1}{m^2}\,\frac{\partial v_2}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{1}{m^2}\,\frac{\partial v_2}{\partial z}\right) + \frac{\mu\omega^2}{c^2}\,v_2 = 0.
\end{aligned}
$$

In the first boundary medium, in which $\varepsilon, \mu, \sigma$ are constants, namely $\varepsilon_0, \mu_0, \sigma_0$, solutions of (7), which can be interpreted as plane waves, are given by the expressions:

$$
\begin{aligned}
u_2 &= a\,e^{-\frac{i\omega}{c} m_0\sqrt{\mu_0}\,(p_0x + q_0z)},\\
v_2 &= b\,e^{-\frac{i\omega}{c} m_0\sqrt{\mu_0}\,(p_0x + q_0z)};
\end{aligned}
$$

$p_0 = \sin\varphi_0$ and $q_0 = \cos\varphi_0$ determine the direction of the wavefront.

This observation leads us to try to find solutions of (7), valid in media in which $\varepsilon$, $\mu$, and $\sigma$ are functions of $z$, which have the special form

$$
(8)\qquad
\begin{aligned}
u_2 &= \left[e^{-\frac{i\omega}{c} m_0\sqrt{\mu_0}\,p_0x}\right] U(z),\\
v_2 &= \left[e^{-\frac{i\omega}{c} m_0\sqrt{\mu_0}\,p_0x}\right] V(z).
\end{aligned}
$$

Actually it can be shown by a consideration of the transient field that the field set up by an harmonic oscillator approaches the form indicated by (8) with increasing time. Since we neglect the transient state, however, we shall show only that the equations (7) can be satisfied by functions of the type (8) and that they represent plane waves travelling through the medium.

If we introduce (8) in equations (7) two ordinary differential equations for U and V are found, namely,

$$\mu \frac{d}{dz}\left(\frac{1}{\mu}\frac{dU}{dz}\right) + \frac{\omega^2}{c^2}\left(m^2\mu - m_o^2\mu_o\, p_o^2\right) U = 0 ,$$

(9)
$$m^2 \frac{d}{dz}\left(\frac{1}{m^2}\frac{dV}{dz}\right) + \frac{\omega^2}{c^2}\left(m^2\mu - m_o^2\mu_o\, p_o^2\right) V = 0 .$$

Upon introducing

(10)
$$M^2 = m^2\mu - m_o^2\mu_o\, p_o^2 ,$$

equations (9) become

$$\mu\left(\frac{U'}{\mu}\right)' + \frac{\omega^2 M^2}{c^2} U = 0 ,$$

(11)
$$m^2\left(\frac{V'}{m^2}\right)' + \frac{\omega^2 M^2}{c^2} V = 0 .$$

Equations (11) are the differential equations upon which our discussion rests.

## 3. Continuity Conditions.

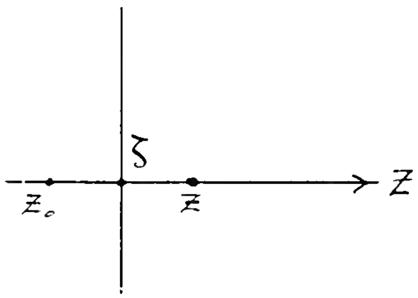


Fig. 3

Let us assume that the functions $\varepsilon, \mu, \sigma$ are discontinuous at a point $z = \zeta$. Then it must be expected that

$$m^2 = \varepsilon - \frac{4\pi\sigma}{\omega} i$$

and

$$M^2 = m^2\mu - m_o^2\mu_o\, p_o^2$$

are also discontinuous functions of z. Yet it follows from (11) that the functions U and V are continuous even at such a point of discontinuity of the medium. Indeed, by integrating (11) from a point $z_o$ to an arbitrary point z (Fig. 3), it follows that

$$\frac{U'(z)}{\mu(z)} - \frac{U'(z_o)}{\mu(z_o)} = -\frac{\omega^2}{c^2}\int_{z_o}^{z} \frac{M^2}{\mu}\, U\, dz$$

$$\frac{V'(z)}{m^2(z)} - \frac{V'(z_o)}{m^2(z_o)} = -\frac{\omega^2}{c^2}\int_{z_o}^{z} \frac{M^2}{m^2}\, V\, dz\,.$$

The indefinite integrals on the right side are continuous even if the integrands have finite discontinuities. It follows that $\frac{U'}{\mu}$ and $\frac{V'}{m^2}$ are continuous.

From

$$U(z) - U(z_o) = \int_{z_o}^{z} \mu \frac{U'}{\mu}\, dz$$

$$V(z) - V(z_o) = \int_{z_o}^{z} m^2 \frac{V'}{m^2}\, dz$$

the continuity of U and V follows by the same argument. Hence we can make the statement: The functions U(z), V(z) and $\frac{1}{\mu}U'(z)$, $\frac{1}{m^2}V'(z)$ are continuous functions of z.

## 4. The Multilayer

In order to justify the method which will be used later to solve the general case in which $\varepsilon, \mu, \sigma$ can vary continuously, we show first how the continuity conditions can be used to construct the solution in case of a multilayer. We assume that the interval $0 < z < \ell$ is divided into r parts of lengths

$$\ell_1, \ell_2, \ldots, \ell_r \text{, respectively,}$$

separated by the points

$$\zeta_0 = 0, \zeta_1, \zeta_2, \ldots, \zeta_{r-1}, \zeta_r = \ell.$$

In every one of the layers it is assumed that $\mathcal{E}, \mu, \sigma$ are constant; $\mathcal{E}_\nu, \mu_\nu, \sigma_\nu$, $\nu = 0, 1, \ldots, r + 1$.

The expressions for the components $u_2$ and $v_2$ of the electromagnetic field in each of these layers is given by (8), where U and V must satisfy (11). From these expressions for $u_2$ and $v_2$ the other field components are obtained by means of (6). Now equations (11) are ordinary differential equations of the second order, and in each of our layers the coefficients are constant. The general solutions are therefore

$$\left.\begin{matrix} U \\ \\ V \end{matrix}\right\} = c_1\, e^{ikz} + c_2 e^{-ikz} ,$$

where k is determined by the values of the coefficients. We may therefore say that the general forms of U and V are known in every layer, namely:

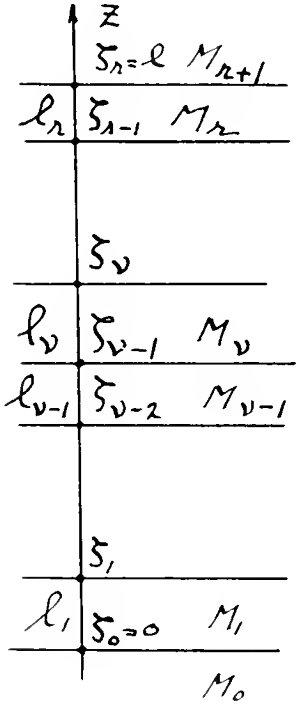


Fig. 4

$$U_{r+1} = a_{r+1} e^{i\frac{\omega}{c} M_{r+1}(z-\zeta_r)} + b_{r+1} e^{-i\frac{\omega}{c} M_{r+1}(z-\zeta_r)}$$

$$U_r = a_r\, e^{i\frac{\omega}{c} M_r(z-\zeta_r)} + b_r e^{-i\frac{\omega}{c} M_r(z-\zeta_r)}$$

- - - - - - - - - - - - - - - - - - - - - - - - - -

$$U_\nu = a_\nu e^{i\frac{\omega}{c} M_\nu(z-\zeta_\nu)} + b_\nu e^{-i\frac{\omega}{c} M_\nu(z-\zeta_\nu)}$$

$$U_{\nu-1} = a_{\nu-1} e^{i\frac{\omega}{c} M_{\nu-1}(z-\zeta_{\nu-1})} + b_{\nu-1} e^{-i\frac{\omega}{c} M_{\nu-1}(z-\zeta_{\nu-1})}$$

- - - - - - - - - - - - - - - - - - - - - - - - - -

$$U_1 = a_1 e^{i\frac{\omega}{c} M_1(z-\zeta_1)} + b_1 e^{-i\frac{\omega}{c} M_1(z-\zeta_1)}$$

$$U_0 = a_0 e^{i\frac{\omega}{c} M_0(z-\zeta_0)} + b_0 e^{-i\frac{\omega}{c} M_0(z-\zeta_0)}$$

Every solution $U_\nu$ thus consists of two parts, the <u>reflected wave</u> $a_\nu e^{i\frac{\omega}{c} M_\nu(z-\zeta_\nu)}$ which travels backward, and the <u>transmitted wave</u>

$b_\nu e^{-i\frac{\omega}{c}M_\nu(z-\zeta_\nu)}$ which travels forward. The amplitude $b_o$ in the first medium, $z < 0$, is the amplitude of the incident wave and has to be considered as given. The quantity $a_o$ gives the complete reflection of the multilayer. The quantity $a_{r+1}$ has the value zero since no wave travels backwards in the upper boundary medium; $b_{r+1}$ on the other hand determines the total transmission through the complete multilayer.

Similar formulas are obtained for the solution V of the second differential equation (11):

$$V_{r+1} = B_{r+1} e^{-i\frac{\omega}{c}M_{r+1}(z-\zeta_r)}$$

- - - - - - - - - - - - - - - - - -

$$V_\nu = A_\nu e^{i\frac{\omega}{c}M_\nu(z-\zeta_\nu)} + B_\nu e^{-i\frac{\omega}{c}M_\nu(z-\zeta_\nu)}$$

- - - - - - - - - - - - - - - - - -

$$V_o = A_o e^{i\frac{\omega}{c}M_o(z-\zeta_o)} + B_o e^{-i\frac{\omega}{c}M_o(z-\zeta_o)}$$

In order to fix the constants in these expressions we note first that $M_\nu$ is given by (10) and (4) with values of $\varepsilon, \mu, \sigma$ belonging to the $\nu$-th layer. The continuity conditions, namely $U(z)$, $V(z)$, $\frac{U'(z)}{\mu(z)}$, and $\frac{V'(z)}{m^2(z)}$ continuous, allow us to set up relations between the coefficients $a_\nu$, $b_\nu$; $A_\nu$, $B_\nu$. The conditions require that at the surface $z = \zeta_r = l$:

$$U_r(l) = U_{r+1}(l), \qquad \frac{1}{\mu_r} U_r'(l) = \frac{1}{\mu_{r+1}} U_{r+1}'(l) \quad ,$$

$$V_r(l) = V_{r+1}(l) \ , \qquad \frac{1}{m_r^2} V_r'(l) = \frac{1}{m_{r+1}^2} V_{r+1}'(l) \ ;$$

and at the surfaces $z = \zeta_{\nu-1}$, $\nu - 1 = 0, 1, \dots, r-1$:

$$U_{\nu-1}(\zeta_{\nu-1}) = U_\nu(\zeta_{\nu-1}) \ , \ \frac{1}{\mu_{\nu-1}} U_{\nu-1}'(\zeta_{\nu-1}) = \frac{1}{\mu_\nu} U_\nu'(\zeta_{\nu-1}) \ ,$$

$$V_{\nu-1}(\zeta_{\nu-1}) = V_\nu(\zeta_{\nu-1}) \ , \ \frac{1}{m_{\nu-1}^2} V'_{\nu-1}(\zeta_{\nu-1}) = \frac{1}{m_\nu^2} V_\nu'(\zeta_{\nu-1}) \ .$$

These equations lead to the following relations among the coefficients $a_\nu$, $b_\nu$; $A_\nu$, $B_\nu$:

(12a)
$$a_r + b_r = b_{r+1}$$
$$\frac{M_r}{\mu_r}(a_r - b_r) = -\frac{M_{r+1}}{\mu_{r+1}} b_{r+1},$$

- - - - - - - - - - - - - - - - -

(12b)
$$a_{\nu-1} + b_{\nu-1} = a_\nu e^{-i\frac{\omega}{c}M_\nu \ell_\nu} + b_\nu e^{i\frac{\omega}{c}M_\nu \ell_\nu},$$
$$\frac{M_{\nu-1}}{\mu_{\nu-1}}(a_{\nu-1} - b_{\nu-1}) = \frac{M_\nu}{\mu_\nu}\left(a_\nu e^{-i\frac{\omega}{c}M_\nu \ell_\nu} - b_\nu e^{i\frac{\omega}{c}M_\nu \ell_\nu}\right);$$

and

(13a)
$$A_r + B_r = B_{r+1}$$
$$\frac{M_r}{m_r^2}(A_r - B_r) = -\frac{M_{r+1}}{m_{r+1}^2} B_{r+1},$$

- - - - - - - - - - - - - - - - -

(13b)
$$A_{\nu-1} + B_{\nu-1} = A_\nu e^{-i\frac{\omega}{c}M_\nu \ell_\nu} + B_\nu e^{i\frac{\omega}{c}M_\nu \ell_\nu},$$
$$\frac{M_{\nu-1}}{m_{\nu-1}^2}(A_{\nu-1} - B_{\nu-1}) = \frac{M_\nu}{m_\nu^2}\left(A_\nu e^{-i\frac{\omega}{c}M_\nu \ell_\nu} - B_\nu e^{i\frac{\omega}{c}M_\nu \ell_\nu}\right).$$

Thus we are led to a system of linear recursion formulas for $a_\nu$, $b_\nu$ and $A_\nu B_\nu$, the solution of which has to satisfy the boundary conditions,

(14)
$$a_{r+1} = 0, \qquad b_o \text{ given},$$
$$A_{r+1} = 0, \qquad B_o \text{ given}.$$

We must solve this system for $a_\nu$, $b_\nu$ and $A_\nu$, $B_\nu$, in terms of known values, i.e., in terms of $b_o$, $B_o$, $M_\nu$, and $\mu_\nu$.

In order to solve this problem we proceed as follows: We write equations (12b) and (13b) in terms of trigonometric functions using the relations, $e^{ik} = \cos k + i \sin k$ and $e^{-ik} = \cos k - i \sin k$:

$$a_{\nu-1} + b_{\nu-1} = (a_\nu + b_\nu) \cos M_\nu T_\nu + i(b_\nu - a_\nu) \sin M_\nu T_\nu ,$$

(15)

$$\frac{M_{\nu-1}}{\mu_{\nu-1}} (a_{\nu-1} - b_{\nu-1}) = \frac{M_\nu}{\mu_\nu} \left[ (a_\nu - b_\nu) \cos M_\nu T_\nu - i\,(a_\nu + b_\nu) \sin M_\nu T_\nu \right];$$

and

$$A_{\nu-1} + B_{\nu-1} = (A_\nu + B_\nu) \cos M_\nu T_\nu + i\,(B_\nu - A_\nu) \sin M_\nu T_\nu ,$$

(16) $$\frac{M_{\nu-1}}{m^2_{\nu-1}} (A_{\nu-1} - B_{\nu-1}) = \frac{M_\nu}{m^2_\nu} \quad (A_\nu - B_\nu) \cos M_\nu T_\nu - i\,(A_\nu + B_\nu) \sin M_\nu T_\nu \quad .$$

The quantity $T_\nu$ is defined by

(17) $$T_\nu = \frac{\omega}{c} \ell_\nu = 2\pi \left(\frac{\ell_\nu}{\lambda}\right) \quad .$$

We introduce the auxiliary quantities

(18)
$$\theta_\nu = -i \frac{M_\nu}{\mu_\nu} \frac{a_\nu - b_\nu}{a_\nu + b_\nu} \quad ,$$

$$\Theta_\nu = -i \frac{M_\nu}{m^2_\nu} \frac{A_\nu - B_\nu}{A_\nu + B_\nu} \quad .$$

By dividing the two equations (15) and (16) respectively, and the two equations (12a) and (13a) respectively, we obtain the following two non-linear recursion formulae for $\theta$ and $\Theta$:

(19)
$$\theta_{\nu-1} = \frac{\theta_\nu - \frac{M_\nu}{\mu_\nu} \tan M_\nu T_\nu}{1 + \theta_\nu \frac{\mu_\nu}{M_\nu} \tan M_\nu T_\nu} , \qquad \theta_r = i \frac{M_{r+1}}{\mu_{r+1}} ;$$

$$\Theta_{\nu-1} = \frac{\Theta_\nu - \frac{M_\nu}{m^2_\nu} \tan M_\nu T_\nu}{1 + \Theta_\nu \frac{m^2_\nu}{M_\nu} \tan M_\nu T_\nu} , \qquad \Theta_r = i \frac{M_{r+1}}{m^2_{r+1}} \quad .$$

Since $\theta_r$ and $\Theta_r$ are known quantities it is possible by successive applications of these recursion formulae to determine the two sets of quantities,

$$\theta_o, \theta_1, \ldots, \theta_r,$$

$$\Theta_o, \Theta_1, \ldots, \Theta_r .$$

In order to find the amplitudes $a_\nu$, $b_\nu$; $A_\nu$, $B_\nu$ we proceed as follows. Writing the first of (18) in the form

$$-i(a_\nu - b_\nu) = \frac{\mu_\nu}{M_\nu}(a_\nu + b_\nu)\ \theta_\nu, \tag{20}$$

we substitute for $i(b_\nu - a_\nu)$ in the first of (15) and obtain

$$\frac{a_{\nu-1} + b_{\nu-1}}{a_\nu + b_\nu} = \cos M_\nu T_\nu + \frac{\mu_\nu}{M_\nu}\theta_\nu \sin M_\nu T_\nu .$$

By repeated applications of this result we get

$$a_\nu + b_\nu = (a_o + b_o) \prod_{\alpha=1}^{\nu} \frac{1}{\cos M_\alpha T_\alpha + \frac{\mu_\alpha}{M_\alpha}\theta_\alpha \sin M_\alpha T_\alpha} \tag{21}$$

Solving (18) for $\frac{a_o}{b_o}$, we write

$$\frac{a_o}{b_o} = \frac{1 + i\frac{\mu_o}{M_o}\theta_o}{1 - i\frac{\mu_o}{M_o}\theta_o}, \tag{22}$$

and thus, adding 1 to both sides,

$$a_o + b_o = \frac{2b_o}{1 - i\frac{\mu_o}{M_o}\theta_o} .$$

Hence, substituting for $a_o + b_o$ in (21),

$$a_\nu + b_\nu = \frac{2b_o}{1 - i\frac{\mu_o}{M_o}\theta_o} \prod_{\alpha=1}^{\nu} \frac{1}{\cos M_\alpha T_\alpha + \frac{\mu_\alpha}{M_\alpha}\theta_\alpha \sin M_\alpha T_\alpha} \tag{23}$$

Furthermore, using (20),

$$(24) \qquad a_\nu - b_\nu = \frac{2b_o \; i \frac{\mu_\nu}{M_\nu} \theta_\nu}{1 - i \frac{\mu_o}{M_o} \theta_o} \prod_{\alpha=1}^{\nu} \frac{1}{\cos M_\alpha T_\alpha + \frac{\mu_\alpha}{M_\alpha} \theta_\alpha \sin M_\alpha T_\alpha}$$

Adding and subtracting (23) and (24) and dividing by $2b_o$,

$$(25) \qquad \frac{a_\nu}{b_o} = \frac{1 + i \frac{\mu_\nu}{M_\nu} \theta_\nu}{1 - i \frac{\mu_o}{M_o} \theta_o} \prod_{\alpha=1}^{\nu} \frac{1}{\cos M_\alpha T_\alpha + \frac{\mu_\alpha}{M_\alpha} \theta_\alpha \sin M_\alpha T_\alpha}$$

$$\frac{b_\nu}{b_o} = \frac{1 - i \frac{\mu_\nu}{M_\nu} \theta_\nu}{1 - i \frac{\mu_o}{M_o} \theta_o} \prod_{\alpha=1}^{\nu} \frac{1}{\cos M_\alpha T_\alpha + \frac{\mu_\alpha}{M_\alpha} \theta_\alpha \sin M_\alpha T_\alpha}$$

Similar formulae can be found for $A_\nu$ and $B_\nu$ simply by replacing in (25) the quantity $\theta_\nu$ by $\Theta_\nu$, $\theta_o$ by $\Theta_o$, and $\mu_\nu$ by $m_\nu^2$ :

$$(26) \qquad \frac{A_\nu}{B_o} = \frac{1 + i \frac{m_\nu^2}{M_\nu} \Theta_\nu}{1 - i \frac{m_o^2}{M_o} \Theta_o} \prod_{\alpha=1}^{\nu} \frac{1}{\cos M_\alpha T_\alpha + \frac{m_\alpha^2}{M_\alpha} \Theta_\alpha \sin M_\alpha T_\alpha}$$

$$\frac{B_\nu}{B_o} = \frac{1 - i \frac{m_\nu^2}{M_\nu} \Theta_\nu}{1 - i \frac{m_o^2}{M_o} \Theta_o} \prod_{\alpha=1}^{\nu} \frac{1}{\cos M_\alpha T_\alpha + \frac{m_\alpha^2}{M_\alpha} \Theta_\alpha \sin M_\alpha T_\alpha}$$

<u>Formulae (25) and (26) express the solution of the problem of the multi-layer in terms of two auxiliary sets of quantities $\theta_\nu$ and $\Theta_\nu$ which satisfy the recursion formulae (19).</u>

It may be observed that by the use of (22) we obtain $a_o$ which determines the complete reflection of the electric field by the multilayer. Similarly, by the use of (25) together with the relation $b_{r+1} = a_r + b_r$,

we obtain $b_{r+1}$ which determines the transmission through the multilayer, $A_o$ and $B_{r+1}$ can be found from the corresponding formulae for the magnetic field.

## 5. Interpretation as Refracted Plane Waves.

The component $E_2$ of the electric vector in the $\nu$-th layer is given by the following expression, on account of (2) and (8) and using the solution of the first of (11):

$$E_2 = e^{i\omega\left[t - \frac{m_o\sqrt{\mu_o}}{c} p_o x\right]} \left\{ a_\nu e^{i\frac{\omega}{c} M_\nu (z-\zeta_\nu)} + b_\nu e^{-i\frac{\omega}{c} M_\nu (z-\zeta_\nu)} \right\}.$$

The right side can obviously be interpreted as the superposition of two plane waves. Let us introduce $p_\nu$ and $q_\nu$ by the definitions:

$$(27)\qquad \begin{aligned} m_\nu\sqrt{\mu_\nu}\, p_\nu &= m_o\sqrt{\mu_o}\, p_o \, , \\ m_\nu\sqrt{\mu_\nu}\, q_\nu &= M_\nu = \sqrt{m_\nu^2\mu_\nu - m_o^2\mu_o\, p_o^2} \, . \end{aligned}$$

It follows that

$$q_\nu = \sqrt{1 - p_\nu^2} \, ,$$

and thus we may interpret $p_\nu$, $q_\nu$ as the direction cosines of the wave normal, whose inclination to the z-axis is denoted by $\varphi_\nu$; that is,

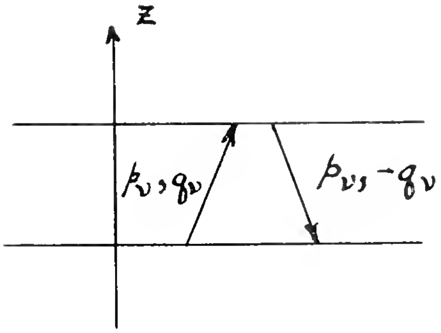
$$p_\nu = \sin\varphi_\nu \, ,$$

$$q_\nu = \cos\varphi_\nu .$$

Making use of (27), the component $E_2$ becomes

$$E_2 = a_\nu e^{-i\frac{\omega}{c} M_\nu \zeta_\nu} e^{i\omega\left[t - \frac{m_\nu\sqrt{\mu_\nu}}{c}(p_\nu x - q_\nu z)\right]} + b_\nu e^{i\frac{\omega}{c} M_\nu \zeta_\nu} e^{i\omega\left[t - \frac{m_\nu\sqrt{\mu_\nu}}{c}(p_\nu x + q_\nu z)\right]}$$

A similar expression can be written for $H_2$ by replacing $a_\nu$ and $b_\nu$ by $A_\nu$ and $B_\nu$ respectively.




The fact that the reflected wave has the direction $(p_\nu, -q_\nu)$ if the transmitted wave has the direction $(p_\nu, q_\nu)$ represents Snell's law of reflection. The first equation (27) represents Snell's law of refraction:*

$$m_\nu\sqrt{\mu_\nu}\ \sin\varphi_\nu = m_0\sqrt{\mu_0}\ \sin\varphi_0 = \text{const.}$$

The laws of reflection and refraction are thus extended to a multilayer. By using the angles of refraction $\varphi_\nu$ in our layers we may determine the quantities $M_\nu$ by the formula,

$$M_\nu = m_\nu\sqrt{\mu_\nu}\ \cos\varphi_\nu,$$

which follows immediately from the second of (27).

We have now completely solved the problem of transmission and reflection of a plane wave incident on a multilayer consisting of a finite number of parallel, plane, homogeneous layers of isotropic media. The expression for $E_2$ just obtained, and the corresponding expression for $H_2$, together with (6) and (2), determine the field in each layer. The amplitudes of the component waves in each layer are given by (25) and (26). In particular, the direction and amplitude of the plane wave transmitted through the entire multilayer can be computed, as well as the direction and amplitude of the plane wave reflected by the multilayer.

## 6. Continuous Variation of $\varepsilon$, $\mu$, and $\sigma$.

We now consider the problem stated at the outset, namely, the propagation of a plane wave incident from below (Fig. 1) on a layer bounded by parallel planes, wherein $\varepsilon$, $\mu$, and $\sigma$ vary continuously with the height z. Our problem is that of determining U(z) and V(z) satisfying (11) together with the boundary conditions appropriate to this case.

We think of the layer as consisting of infinitely many, infinitely thin parallel layers each with constant $\varepsilon$, $\mu$, and $\sigma$. We are thus led to form the sets of functions, containing the parameter $\zeta$,

---

* cf. Stratton: Electromagnetic Theory, p 491.

$$\mathfrak{U}(z;\zeta) = a(\zeta)\, e^{i\frac{\omega}{c}M(\zeta)(z-\zeta)} + b(\zeta)\, e^{-i\frac{\omega}{c}M(\zeta)(z-\zeta)},$$

(28)

$$\mathfrak{V}(z;\zeta) = A(\zeta)\, e^{i\frac{\omega}{c}M(\zeta)(z-\zeta)} + B(\zeta)\, e^{-i\frac{\omega}{c}M(\zeta)(z-\zeta)},$$

suggested by the solutions for the multilayer, and to consider that $U(z) = \mathfrak{U}(z;\zeta)$, $V(z) = \mathfrak{V}(z;\zeta)$ for the infinitely thin layer at $z = \zeta$. The functions $U(z)$ and $V(z)$ are thus constructed as envelopes of the two sets of functions $\mathfrak{U}(z;\zeta)$ and $\mathfrak{V}(z;\zeta)$. The former continuity conditions are replaced by the conditions that at $z = \zeta$ the functions $\mathfrak{U}$ and $\mathfrak{V}$ have the same values and derivatives as $U(\zeta)$ and $V(\zeta)$.

Expressed as equations, these conditions are

$$\mathfrak{U}(z;\zeta) = U(\zeta), \qquad \mathfrak{V}(z;\zeta) = V(\zeta),$$

(29)

$$\frac{\partial \mathfrak{U}(z;\zeta)}{\partial z} = U'(\zeta), \qquad \frac{\partial \mathfrak{V}(z;\zeta)}{\partial z} = V'(\zeta).$$

at $z = \zeta$.

The physical interpretation of the formulas (28) and (29) is that the actual propagated wave in the continuously varying layer is constructed as the envelope of a set of <u>plane waves</u>. We may call these plane waves the <u>tangential waves</u> of the solution. Indeed their relation to the solution is comparable with the relation of the tangents of a curve to the curve itself.

We must now obtain the functions U and V and hence the functions $a(z)$, $b(z)$; $A(z)$, $B(z)$, from (28) by means of conditions (29). The quantity M is known through (10) wherein $\varepsilon$, $\mu$, and $\sigma$ are known functions of z in the layer. The application of conditions (29) to (28) yields

$$U(\zeta) = a(\zeta) + b(\zeta), \qquad V(\zeta) = A(\zeta) + B(\zeta),$$

(30)

$$U'(\zeta) = i\,\frac{\omega}{c}M(\zeta)\left[a(\zeta) - b(\zeta)\right], \quad V'(\zeta) = \frac{i\omega}{c}M(\zeta)\left[A(\zeta) - B(\zeta)\right].$$

The equations (30) may be used directly to define the functions a, b and A, B, without any reference to our preceding remarks. Indeed the purpose of those considerations was to justify relations (30), which otherwise would seem rather artificial, and also to justify the interpretation of a, A; b, B as amplitudes of the reflected and transmitted wave, respectively.

From (30) we have

$$a' + b' = \frac{i\omega}{c} M(a-b), \qquad A' + B' = \frac{i\omega}{c} M(A-B).$$

Applying equations (11) to (30),

$$\mu \frac{i\omega}{c} \left(\frac{M(a-b)}{\mu}\right)' + \frac{\omega^2}{c^2} M^2(a+b) = 0, \quad m^2 \frac{i\omega}{c} \left(\frac{M(A-B)}{m^2}\right)' + \frac{\omega^2}{c^2} M^2(A+B) = 0.$$

Thus a, b; A, B satisfy a system of linear differential equations of the first order, namely,

$$(31) \qquad \begin{aligned} a' + b' &= \frac{i\omega}{c} M (a-b), & A' + B' &= \frac{i\omega}{c} M(A-B), \\ \mu \left(\frac{M(a-b)}{\mu}\right)' &= i \frac{\omega}{c} M^2(a+b), & m^2 \left(\frac{M(A-B)}{m^2}\right)' &= \frac{i\omega}{c} M^2(A + B). \end{aligned}$$

The problem is to find a solution a, b; A, B which satisfies the boundary conditions:

$$a(\ell + 0) = 0, \qquad A(\ell + 0) = 0,$$

$b(-0)$ given $\qquad B(-0)$ given.

The notation $\ell + 0$ indicates that the value on the right of $\ell$ is meant if the functions should have discontinuities at $z = \ell$. Similarly

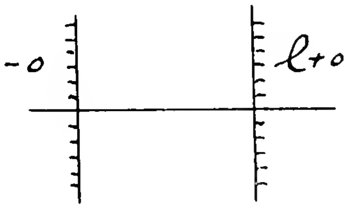



Fig. 6

the notation $-0$ means the value on the left side of 0 if discontinuities occur at $z = 0$.

As in the case of the multilayer, we introduce two auxiliary functions:

$$(32) \qquad \theta(z) = -i\,\frac{M}{\mu}\,\frac{a-b}{a+b}\,, \qquad\qquad \Theta(z) = -i\,\frac{M}{m^2}\,\frac{A-B}{A+B}\,,$$

suggested by (18). By differentiation,

$$\theta' = -i\,\frac{1}{a+b}\left[\frac{M}{\mu}(a-b)\right]' + i\,\frac{M(a-b)}{\mu}\,\frac{a'+b'}{(a+b)^2}\,.$$

Hence on account of (31) and recalling that $\frac{\omega}{c} = \frac{2\pi}{\lambda}$,

$$\theta' = \frac{\omega}{c}\left(\frac{M^2}{\mu} + \mu\theta^2\right) = \frac{2\pi}{\lambda}\left(\frac{M^2}{\mu} + \mu\theta^2\right)\,.$$

Similarly,

$$\Theta' = \frac{\omega}{c}\left(\frac{M^2}{m^2} + m^2\Theta^2\right) = \frac{2\pi}{\lambda}\left(\frac{M^2}{m^2} + m^2\Theta^2\right)\,.$$

Thus far we have the following result: <u>The functions $\theta(z)$ and $\Theta(z)$ satisfy differential equations of Riccati's type:</u>

$$(33) \qquad \begin{aligned} \theta' &= \frac{2\pi}{\lambda}\left(\frac{M^2}{\mu} + \mu\theta^2\right), \\ \Theta' &= \frac{2\pi}{\lambda}\left(\frac{M^2}{m^2} + m^2\Theta^2\right), \end{aligned}$$

<u>and are uniquely determined by the boundary conditions</u>

$$(33') \qquad \theta(\ell) = i\,\frac{M(\ell+0)}{\mu(\ell+0)}\,, \qquad \Theta(\ell) = i\,\frac{M(\ell+0)}{m^2(\ell+0)}\,.$$

The boundary conditions follow directly from the fact that the functions $\theta(z)$ and $\Theta(z)$ are continuous functions of $z$ even if $\varepsilon, \mu, \sigma$ have finite discontinuities. This continuity follows, as in the former case of $U(z)$ and $V(z)$, by integration of the differential equations:

$$\theta(z) - \theta(z_0) = \frac{2\pi}{\lambda}\int_{z_0}^{z}\left(\frac{M^2}{\mu} + \mu\theta^2\right) dz,$$

$$\Theta(z) - \Theta(z_0) = \frac{2\pi}{\lambda}\int_{z_0}^{z}\left(\frac{M^2}{m^2} + m^2\Theta^2\right) dz.$$

Hence, since $a(\ell + 0) = 0$, and $A(\ell + 0) = 0$, we have from (32),

$$\theta(\ell) = \theta(\ell + 0) = i\,\frac{M(\ell + 0)}{\mu(\ell + 0)} ,$$

$$\Theta(\ell) = \Theta(\ell + 0) = i\,\frac{M(\ell + 0)}{m^2(\ell + 0)} .$$

With the functions $\theta$ and $\Theta$ we are in the position to find the amplitudes a, b; A, B by quadratures. We introduce in the first of (31) the expression, from (32),

$$iM(a-b) = -\mu\theta(z)(a+b),$$

and obtain

$$\frac{a'+b'}{a+b} = -\frac{\omega}{c}\mu\theta(z) .$$

Integrating,

$$a + b = (a_0 + b_0)\, e^{-\frac{\omega}{c}\int_0^z \mu\theta\, dz} ,$$

where the subscript 0 has the same meaning as in section 4, i.e. the value for $z<0$. Putting $z < 0$ in (32) and solving for $a_0 + b_0$,

$$a_0 + b_0 = \frac{2b_0}{1-i\frac{\mu_0}{M_0}\theta_0} ,$$

and hence

$$a + b = \frac{2b_0}{1-i\frac{\mu_0}{M_0}\theta_0} \cdot e^{-\frac{\omega}{c}\int_0^z \mu\theta\, dz}.$$

Solving (32) for a-b and inserting the value of a+b just found,

$$a-b = \frac{i\mu}{M}\,\Theta\,\frac{2b_o}{1-i\frac{\mu_o}{M_o}\Theta_o}\cdot e^{-\frac{\omega}{c}\int_0^z \mu\Theta\,dz}\,.$$

Hence, solving these two equations for a and b, we have the result: <u>If $\Theta(z)$ and $\ominus(z)$ are the solutions of the problem (33) then the solutions a, b; A, B of (31) are given by the formulae:</u>

$$\frac{a}{b_o} = \frac{1+i\frac{\mu}{M}\Theta}{1-i\frac{\mu_o}{M_o}\Theta_o}\cdot e^{-\frac{\omega}{c}\int_0^z \mu\Theta\,dz},$$

$$(34)\qquad \frac{b}{b_o} = \frac{1-i\frac{\mu}{M}\Theta}{1-i\frac{\mu_o}{M_o}\Theta_o}\cdot e^{-\frac{\omega}{c}\int_0^z \mu\Theta\,dz},$$

$$\frac{A}{B_o} = \frac{1+i\frac{m^2}{M}\ominus}{1-i\frac{m_o^2}{M_o}\ominus_o}\cdot e^{-\frac{\omega}{c}\int_0^z m^2\ominus\,dz},$$

$$(35)\qquad \frac{B}{B_o} = \frac{1-i\frac{m^2}{M}\ominus}{1-i\frac{m_o^2}{M_o}\ominus_o}\cdot e^{-\frac{\omega}{c}\int_0^z m^2\ominus\,dz}.$$

The formulae (35) are obtained from (34) by replacing $\mu$ by $m^2$ and $\Theta$ by $\ominus$. If these values of a, b; A, B are inserted in the first of equations (30) and the resulting values of U and V are put in (8) we shall have expressions for $u_2$ and $v_2$.

It is to be noted that formulas (34) and (35) hold for $z < 0$ as well as for $z > 0$. In particular, a(-0), b(-0), A(-0), and B(-0) can be computed by letting z approach zero through negative values.

## 7. The Electromagnetic Field in the Continuous Layer.

The preceding considerations and results make it possible to construct the electromagnetic wave which travels through the medium. We have seen that this wave can be represented by vectors E and H having the form

$$E = (u_1, u_2, u_3) \cdot e^{i\omega t},$$

$$H = (v_1, v_2, v_3) \cdot e^{i\omega t}.$$

The vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ depend only on x and z. We have found the components $u_2$ and $v_2$ and represented them as envelopes of plane tangential waves:

$$\mathcal{U}_2(x,z;\zeta) = a(\zeta) e^{-i\frac{\omega}{c}\left[m_o\sqrt{\mu_o}\, p_o x - M(\zeta)(z-\zeta)\right]} + b(\zeta) e^{-i\frac{\omega}{c}\left[m_o\sqrt{\mu_o}\, p_o x + M(\zeta)(z-\zeta)\right]},$$

$$\mathcal{V}_2(x,z;\zeta) = A(\zeta) e^{-i\frac{\omega}{c}\left[M_o\sqrt{\mu_o}\, p_o x - M(\zeta)(z-\zeta)\right]} + B(\zeta) e^{-i\frac{\omega}{c}\left[m_o\sqrt{\mu_o}\, p_o x + M(\zeta)(z-\zeta)\right]},$$

where a, b; A, B are given by (34) and (35).

As in the case of the multilayer (section 5) we introduce the direction cosines of these plane waves:

$$m(\zeta)\sqrt{\mu(\zeta)}\ \ p(\zeta) = m_o\sqrt{\mu_o}\, p_o\ ,$$

$$m(\zeta)\sqrt{\mu(\zeta)}\ \ q(\zeta) = M(\zeta) = \sqrt{m^2(\zeta)\mu(\zeta) - m_o^2\mu_o p_o^2}\ .$$

Since

$$q = \sqrt{1-p^2}\ ,$$

we may interpret

$$p = \sin\varphi,$$

$$q = \cos\varphi,$$

as the continuously varying direction cosines of the wave normal, and thus recognize the relation,

$$m\sqrt{\mu}\ p = m\sqrt{\mu}\ \sin\varphi = m_o\sqrt{\mu_o}\ \ \sin\varphi_o.$$

as Snell's law of refraction, extended to media with continuously varying $\varepsilon$, $\mu$, and $\sigma$.

With this notation:

$$\mathcal{U}_2(x,z;\zeta)=a(\zeta)\, e^{-i\frac{\omega}{c} m\sqrt{\mu}\left[px - q(z-\zeta)\right]} + b(\zeta)\, e^{-i\frac{\omega}{c} m\sqrt{\mu}\left[px + q(z-\zeta)\right]},$$

$$\mathcal{V}_2(x,z;\zeta)=A(\zeta)\, e^{-i\frac{\omega}{c} m\sqrt{\mu}\left[px - q(z-\zeta)\right]} + B(\zeta)\, e^{-i\frac{\omega}{c} m\sqrt{\mu}\left[px + q(z-\zeta)\right]},$$

in which a, b; A, B are given in (34) and (35). We now represent not only the components $u_2$ and $v_2$ but the whole vector $(u_1,u_2,u_3)$ and $(v_1,v_2,v_3)$ as envelopes of tangential plane waves:

$$(\mathcal{U}_1,\mathcal{U}_2,\mathcal{U}_3)=(a_1,a_2,a_3)e^{-i\frac{\omega}{c} m\sqrt{\mu}\left[px-q(z-\zeta)\right]}+(b_1,b_2,b_3)e^{-i\frac{\omega}{c} m\sqrt{\mu}\left[px + q(z-\zeta)\right]},$$

$$(\mathcal{V}_1,\mathcal{V}_2,\mathcal{V}_3)=(A_1,A_2,A_3)e^{-i\frac{\omega}{c} m\sqrt{\mu}\left[px-q(z-\zeta)\right]} + (B_1,B_2,B_3)e^{-i\frac{\omega}{c} m\sqrt{\mu}\left[px + q(z-\zeta)\right]},$$

and the remaining problem is to find the four vectors,

$(a_1,a_2,a_3)$ amplitude of reflected electric field,
$(b_1,b_2,b_3)$ amplitude of transmitted electric field,
$(A_1,A_2,A_3)$ amplitude of reflected magnetic field,
$(B_1,B_2,B_3)$ amplitude of transmitted magnetic field.

We have, of course, the relations

$$a_2 = a\,, \qquad A_2 = A,$$
$$b_2 = b\,, \qquad B_2 = B.$$

In order to find the remaining components $a_1, a_3$; $b_1, b_3$; $A_1, A_3$; $B_1 B_3$ we make use of the relations (6) which lead to similar relations between the plane tangential waves:

$$\mathfrak{V}_1 = i \frac{c}{m^2\omega} \frac{\partial \mathfrak{W}_2}{\partial z}, \qquad \mathfrak{W}_1 = -i \frac{c}{\mu\omega} \frac{\partial \mathfrak{V}_2}{\partial z},$$

(36)

$$\mathfrak{V}_3 = -i \frac{c}{m^2\omega} \frac{\partial \mathfrak{W}_2}{\partial x}, \qquad \mathfrak{W}_3 = i \frac{c}{\mu\omega} \frac{\partial \mathfrak{V}_2}{\partial x}.$$

From these equations the desired expressions for $a_1, a_3$; $b_1, b_3$ etc. can be found. For example,

$$\mathfrak{V}_1(x,z;\zeta) = a_1(\zeta) e^{-\frac{i\omega}{c} m\sqrt{\mu}\left[px - q(z-\zeta)\right]} + b_1(\zeta) e^{-i\frac{\omega}{c} m\sqrt{\mu}\left[px + q(z-\zeta)\right]}$$

and

$$i \frac{c}{m^2\omega} \frac{\partial \mathfrak{W}_2}{\partial z} = \frac{\sqrt{\mu}}{m} q \left\{ -A_2(\zeta) e^{-\frac{i\omega}{c} m\sqrt{\mu}\left[px - q(z-\zeta)\right]} + B_2 e^{-\frac{i\omega}{c} m\sqrt{\mu}\left[px + q(z-\zeta)\right]} \right\}$$

Applying the first of equations (36), equating coefficients of the two exponential functions, and setting $\zeta = z$, we obtain the values of $a_1$ and $b_1$. Making similar use of the four equations (36) we find

$$a_1 = -\frac{\sqrt{\mu}}{m} q A \qquad b_1 = \frac{\sqrt{\mu}}{m} q B$$

(37)

$$a_2 = a \qquad b_2 = b$$

$$a_3 = -\frac{\sqrt{\mu}}{m} p A \qquad b_3 = -\frac{\sqrt{\mu}}{m} p B$$

$$A_1 = \frac{m}{\sqrt{\mu}} q a \qquad B_1 = -\frac{m}{\sqrt{\mu}} q b$$

(38)

$$A_2 = A \qquad B_2 = B$$

$$A_3 = \frac{m}{\sqrt{\mu}} p a \qquad B_3 = \frac{m}{\sqrt{\mu}} p b$$

It may be remarked that the components $u_1, u_3$, $v_1$ and $v_3$, can also be obtained by inserting the values of $u_2$ and $v_2$, found as described at the end of section 6, in equations (6). This method, however, will not readily give us the quantities $a_1, a_3; b_1, b_3$; $A_1, A_3$; $B_1, B_3$.

We conclude from equations (37) and (38):

1) The two reflected vectors $(a_1, a_2, a_3)$ and $(A_1, A_2, A_3)$ are perpendicular to the wave normal $(p, 0, -q)$ of the reflected wave. In fact: The two scalar products $a_1 p + a_2 . 0 + a_3(-q)$ and $A_1 p + A_2 . 0 + A_3(-q)$ are zero.

2) The two transmitted vectors $(b_1, b_2, b_3)$ and $(B_1, B_2, B_3)$ are perpendicular to the wave normal $(p, 0, q)$ of the transmitted wave.

These two statements establish the transverse character of the electromagnetic waves.

3) The electric and magnetic vectors are perpendicular to each other.

Indeed: $(a_1, a_2, a_3)$ is normal to $(A_1, A_2, A_3)$,

and $(b_1, b_2, b_3)$ is normal to $(B_1, B_2, B_3)$.

These results suggest the possibility of determining the electromagnetic vectors not with respect to the original fixed coordinate system, but with respect to a system which changes with the direction of the wave front. We choose the wave normal N as z'-axis and the original y-direction as the y'-direction; the x'-direction is then determined. It is customary to call the y'-component of a vector the normal or S-component since the y'-direction is normal to the plane of incidence. The x'-component is called the parallel or p-component since the x'-direction is parallel to the plane of incidence. In Fig. 7 and Fig. 8 the definitions are illustrated for the cases of the transmitted and reflected waves, respectively. These components are obtained as follows: For the reflected wave the direction cosines of the normal are $(p, 0, -q)$; hence the direction cosines of the p-component are $(-q, 0, -p)$. Similarly, for the transmitted wave the direction cosine of the p-component are $(q, 0, p)$. Hence,

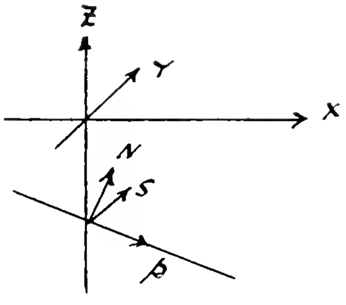



Fig. 7

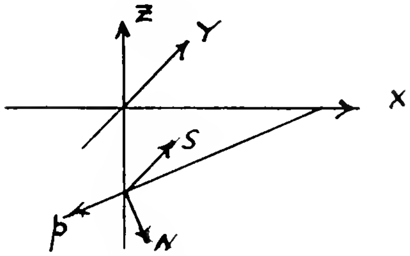



Fig. 8

$$
(39)\qquad
\begin{aligned}
a_p &= -(qa_1 + pa_3) \\
a_s &= a_2 \\
b_p &= qb_1 - pb_3 \\
b_s &= b_2 \\
A_p &= -(qA_1 + pA_3) \\
A_s &= A_2 \\
B_p &= qB_1 - \rho B_3 \\
B_s &= B_2
\end{aligned}
$$

Substituting from (37) and (38) in (39), the following expressions are found:

$$
(40)\qquad
\begin{aligned}
a_p &= +\sqrt{\frac{\mu}{m}}\, A\,, & b_p &= +\sqrt{\frac{\mu}{m}}\, B\,, \\
a_s &= a\,, & b_s &= b\,.
\end{aligned}
$$

$$
(41)\qquad
\begin{aligned}
A_p &= -\frac{m}{\sqrt{\mu}}\, a, & B_p &= -\frac{m}{\sqrt{\mu}}\, b, \\
A_s &= A\,, & B_s &= B\,.
\end{aligned}
$$

If we denote by $b_p(0)$, $b_s(0)$ the values of $b_p$ and $b_s$, respectively, for $z < 0$, it follows that for the electric field

$$
(42)\qquad
\begin{aligned}
\frac{a_p}{b_p(0)} &= \sqrt{\frac{\mu}{\mu_o}}\,\frac{m_o}{m}\,\frac{A}{B_o}\,, & \frac{b_p}{b_p(0)} &= \sqrt{\frac{\mu}{\mu_o}}\,\frac{m_o}{m}\,\frac{B}{B_o}\,, \\
\frac{a_s}{b_s(0)} &= \frac{a}{b_o}\,, & \frac{b_s}{b_s(0)} &= \frac{b}{b_o}\,.
\end{aligned}
$$

and for the magnetic field

$$\frac{A_p}{B_p(0)} = \sqrt{\frac{\mu_o}{\mu}}\,\frac{m}{m_o}\,\frac{a}{b_o}\,, \qquad \frac{B_p}{B_p(0)} = \sqrt{\frac{\mu_o}{\mu}}\,\frac{m}{m_o}\,\frac{b}{b_o}\,,$$

(43)

$$\frac{A_s}{B_s(0)} = \frac{A}{B_o}\,, \qquad \frac{B_s}{B_s(0)} = \frac{B_s}{B_o}\,.$$

The ratios $\frac{a}{b_o}$, $\frac{b}{b_o}$; $\frac{A}{B_o}$, $\frac{B}{B_o}$ are determined by the equations (34) and (35). Our formulae (42) and (43) thus permit us to determine the electromagnetic field if the amplitudes in the incident wave $b_p(0)$ and $b_s(0)$ of the electric vector and $B_p(0)$ and $B_s(0)$ of the magnetic vector are given.

## 8. Reflection and Transmission of the Continuous Layer.

The reflection and transmission of the whole system which is illustrated in Fig. 1 is characterized by the reflected wave in the medium $\varepsilon_o, \mu_o, \sigma_o$ and by the transmitted wave in the medium $\varepsilon_1, \mu_1, \sigma_1$. In the following we shall consider only the electric vector in detail. The treatment and results for the magnetic vector are exactly similar.

We introduce the Reflection Coefficients, which are ratios of reflected amplitudes to incident amplitudes,

(44)

$$R_p = \frac{a_p(-0)}{b_p(-0)}\,,$$

$$R_s = \frac{a_s(-0)}{b_s(-0)}\,,$$

and the Transmission Coefficients, which are ratios of transmitted amplitudes to incident amplitudes,

(45)

$$D_p = \frac{b_p(\ell+0)}{b_p(-0)}\,,$$

$$D_s = \frac{b_s(\ell+0)}{b_s(-0)}\,.$$

The formulas (42) in conjunction with (34) and (35) give the following result:

Let $\theta(z)$ and $\Theta(z)$ be the solutions of the Riccati equations

$$(33)\qquad \left.\begin{aligned} \theta' &= \frac{2\pi}{\lambda}\left(\frac{M^2}{\mu} + \mu\theta^2\right), \\ \Theta' &= \frac{2\pi}{\lambda}\left(\frac{M^2}{m^2} + m^2\Theta^2\right), \end{aligned}\right\} \quad \begin{aligned} M^2 &= m^2\mu - m_o^2\mu_o \sin^2\varphi_o. \\ m &= n - ink. \end{aligned}$$

which satisfy the boundary conditions,

$$(33')\qquad \theta(\ell) = i\frac{M_1}{\mu_1} = \frac{M(\ell+0)}{\mu(\ell+0)}, \quad \Theta(\ell) = i\frac{M_1}{m_1^2} = \frac{M(\ell+0)}{\mu(\ell+0)} .$$

Then the reflection and transmission of the layer system is given by the expressions:

$$(46)\qquad R_p = \frac{1 + i\frac{m_o^2}{M_o}\Theta_o}{1 - i\frac{m_o^2}{M_o}\Theta_o} . \qquad R_s = \frac{1 + i\frac{\mu_o}{M_o}\theta_o}{1 - i\frac{\mu_o}{M_o}\theta_o} ,$$

and

$$(47)\qquad \begin{aligned} D_p &= \frac{2\sqrt{\frac{\mu_1}{\mu_o}}\,\frac{m_o}{m_1}}{1 - i\frac{m_o^2}{M_o}\Theta_o}\; e^{-\frac{2\pi}{\lambda}\int_0^\ell m^2\Theta(z)dz} , \\ D_s &= \frac{2}{1 - i\frac{\mu_o}{M_o}\theta_o}\; e^{-\frac{2\pi}{\lambda}\int_0^\ell \mu\theta(z)dz} . \end{aligned}$$

Equations (46) are obtained by using (44) with (42) and substituting from (34) and (35). Equations (47) are obtained by using (45) with (42), substituting from (34) and (35), and using the boundary conditions (33').

## 9. Reflection and Transmission of the Multilayer.

The two Riccati equations (33) are replaced by two recursion formulas (19) in the case of the multilayer. Let $\theta$ and $\Theta$ be the solutions of the two recursion formulas

$$(19) \qquad \theta_{\nu-1} = \frac{\theta_\nu - \frac{M_\nu}{\mu_\nu} \tan M_\nu T_\nu}{1 + \theta_\nu \frac{\mu_\nu}{M_\nu} \tan M_\nu T_\nu}, \qquad \Theta_{\nu-1} = \frac{\Theta_\nu - \frac{M_\nu}{m_\nu^2} \tan M_\nu T_\nu}{1 + \Theta_\nu \frac{m_\nu^2}{M_\nu} \tan M_\nu T_\nu},$$

which satisfy

$$\theta_r = i \frac{M_{r+1}}{\mu_{r+1}}, \qquad \Theta_r = i \frac{M_{r+1}}{m_{r+1}^2}.$$

Again let the reflection and transmission coefficients be defined by (44) and (45), respectively. We note that in any layer $a_s = a_\nu$ and $b_s = b_\nu$. In order to calculate $a_p$ and $b_p$ we first use (6) together with (8) and the expressions for $U(z)$ and $V(z)$ found for the multilayer, thus obtaining the three components of the reflected field and the transmitted field, respectively, in any layer. Applying these steps for the lower boundary layer and using (39) and the relation $M_o = m_o\sqrt{\mu_o}\, q_o$, we obtain

$$a_p(-0) = A_o \frac{\sqrt{\mu_o}}{m_o}, \qquad b_p(-0) = B_o \frac{\sqrt{\mu_o}}{m_o}.$$

Now applying formula (22) and the corresponding formula for $\frac{A_o}{B_o}$, we have

$$(48) \qquad R_p = \frac{1 + i \frac{m_o^2}{M_o} \Theta_o}{1 - i \frac{m_o^2}{M_o} \Theta_o}, \qquad R_s = \frac{1 + i \frac{\mu_o}{M_o} \theta_o}{1 - i \frac{\mu_o}{M_o} \theta_o}.$$

In a similar way we find

$$b_p(l + 0) = B_{r+1} \frac{\sqrt{\mu_{r+1}}}{m_{r+1}}.$$

Making use of the relations $b_{r+1} = a_r + b_r$, $B_{r+1} = A_r + B_r$, and applying (25) and (26), we get

$$D_p = \frac{2\sqrt{\frac{\mu_{r+1}}{\mu_0}}\ \frac{m_o}{m_{r+1}}}{1 - i\frac{m_o^2}{M_o}\Theta_o} \prod_{\nu=1}^{r} \frac{1}{\cos M_\nu T_\nu + \frac{m_\nu^2}{M_\nu}\Theta_\nu \sin M_\nu T_\nu}$$

(49)

$$D_s = \frac{2}{1 - i\frac{\mu_o}{M_o}\Theta_o} \prod_{\nu=1}^{r} \frac{1}{\cos M_\nu T_\nu + \frac{\mu_\nu}{M_\nu}\Theta_\nu \sin M_\nu T_\nu}$$

It is of considerable interest that the preceding formulas for the multilayer, namely (19), (48) and (49) which were just derived on the basis of multilayer theory alone, can be derived from the equations (33), (46) and (47) which are valid for the continuous variation of the substances. Let us, for example, consider the equation

$$\Theta' = \frac{2\pi}{\lambda}\left(\frac{M^2}{\mu} + \mu\Theta^2\right).$$

In the $\nu$-th layer both M and $\mu$ are constants, $M_\nu$ and $\mu_\nu$, and the differential equation can be integrated readily. Let $\Theta_\nu = \Theta(\zeta_\nu)$. We have from (33),

$$\frac{\Theta'}{\frac{M_\nu^2}{\mu_\nu} + \mu_\nu\Theta^2} = \frac{2\pi}{\lambda},$$

and thus

$$\text{arc tg}\,\frac{\mu_\nu}{M_\nu}\,\Theta - \text{arc tg}\,\frac{\mu_\nu}{M_\nu}\,\Theta_\nu = \frac{2\pi}{\lambda}\,M_\nu\,(z-\zeta_\nu).$$

Solving for $\Theta(z)$, we have

$$\Theta(z) = \frac{\Theta_\nu + \frac{M_\nu}{\mu_\nu}\tan\frac{2\pi}{\lambda}M_\nu(z-\zeta_\nu)}{1 - \Theta_\nu\frac{\mu_\nu}{M_\nu}\tan\frac{2\pi}{\lambda}M_\nu(z-\zeta_\nu)},$$

valid in the $\nu$-th layer. But we have remarked already that $\Theta(z)$ must be continuous even in case of discontinuous $\varepsilon, \mu, \sigma$. Hence $\Theta_\nu = \Theta(\zeta_\nu + 0) = \Theta(\zeta_\nu - 0)$. Letting $z = \zeta_{\nu-1}$ in our former formula we must obtain on the left side the quantity $\Theta_{\nu-1}$. The result of this is the recursion formula (19):

$$\Theta_{\nu-1} = \frac{\Theta_\nu - \frac{M_\nu}{\mu_\nu}\tan M_\nu T_\nu}{1 + \Theta_\nu\frac{\mu_\nu}{M_\nu}\tan M_\nu T_\nu}; \qquad T_\nu = \frac{2\pi}{\lambda}\,l_\nu.$$

Similarly, the second recursion formula (19) can be derived.

In order to derive the expressions (49) from (47) we first determine the integral $\frac{2\pi}{\lambda} \int_{\zeta_{\nu-1}}^{\zeta_\nu} \mu\Theta \, dz$. We write

$$\frac{2\pi}{\lambda} \int_{\zeta_{\nu-1}}^{\zeta_\nu} \mu_\nu \Theta \, dz = \mu_\nu \int_{\zeta_{\nu-1}}^{\zeta_\nu} \frac{2\pi}{\lambda} \Theta \, dz = \mu_\nu \int_{\zeta_{\nu-1}}^{\zeta_\nu} \frac{\Theta \, \Theta'}{\frac{M_\nu^2}{\mu_\nu} + \mu_\nu \Theta^2} \, dz$$

since, from (33), $\frac{2\pi}{\lambda} = \dfrac{\Theta'}{\frac{M_\nu^2}{\mu_\nu} + \mu_\nu \Theta^2}$ .

The integral is therefore equal to $\frac{1}{2} \log \dfrac{\frac{M_\nu^2}{\mu_\nu} + \mu_\nu \Theta_\nu^2}{\frac{M_\nu^2}{\mu_\nu} + \mu_\nu \Theta_{\nu-1}^2}$ .

On account of the recursion formula for $\Theta_{\nu-1}$

$$\frac{M_\nu^2}{\mu_\nu} + \mu_\nu \Theta_{\nu-1}^2 = \left( \frac{M_\nu^2}{\mu_\nu} + \mu_\nu \Theta_\nu^2 \right) \frac{1 + \tan^2 M_\nu T_\nu}{\left(1 + \frac{\mu_\nu}{M_\nu} \Theta_\nu \tan M_\nu T_\nu\right)^2} ,$$

and hence

$$\frac{2\pi}{\lambda} \int_{\zeta_{\nu-1}}^{\zeta_\nu} \mu \Theta \, dz = \log \left( \cos M_\nu T_\nu + \frac{\mu_\nu}{M_\nu} \Theta_\nu \sin M_\nu T_\nu \right) .$$

Finally,

$$\frac{2\pi}{\lambda} \int_0^{l} \mu \Theta \, dz = \log \prod_{\nu=1}^{r} \left( \cos M_\nu T_\nu + \frac{\mu_\nu}{M_\nu} \Theta_\nu \sin M_\nu T_\nu \right) .$$

By introducing this in the second equation (47) the second equation (49) is obtained. Similarly the first equation (49) can be derived from the first equation (47).

## 10. The case of Normal Incidence.

The difference between normal and parallel components loses point if the incident wave front is parallel to the xy-plane. The parallel component of the transmitted wave coincides with the x-component in the original coordinate system. The parallel component of the reflected wave coincides with the negative x-component of the original system. Consequently we expect the following relations in this case:

$$R_p = -R_s, \qquad D_p = D_s \, . \tag{50}$$

We prove these relations only for the case of continuous variation since we have seen that the multilayer case can be considered as a special case.

In case of normal incidence we have $p_o = 0$ and $M = m\sqrt{\mu}$; hence the Riccati equations are

$$\theta' = \frac{2\pi}{\lambda}(m^2 + \mu\theta^2) \, , \qquad \Theta' = \frac{2\pi}{\lambda}(\mu + m^2\Theta^2) \, , \tag{51}$$

and the boundary conditions are

$$\theta(\ell) = i\frac{m_1}{\sqrt{\mu_1}} \quad , \qquad \Theta(\ell) = i\frac{\sqrt{\mu_1}}{m_1} \, .$$

If we write the second equation (51) in the form

$$-\left(\frac{1}{\Theta}\right)' = \frac{2\pi}{\lambda}\left(\frac{\mu}{\Theta^2} + m^2\right) ,$$

and the second boundary condition

$$-\frac{1}{\Theta} = i\frac{m_1}{\sqrt{\mu_1}} \, ,$$

we readily recognize the relation

$$\Theta = -\frac{1}{\theta} \, .$$

We thus introduce $\Theta_o = -\,{}^1/\theta_o$ in the expression (46) for $R_p$ and obtain

$$R_p = \frac{1 - \frac{i\,m_o}{\sqrt{\mu_o}}\,\frac{1}{\theta_o}}{1 + \frac{i\,m_o}{\sqrt{\mu_o}}\,\frac{1}{\theta_o}} = -\,\frac{1 + i\,\frac{\sqrt{\mu_o}}{m_o}\,\theta_o}{1 - i\,\frac{\sqrt{\mu_o}}{m_o}\,\theta_o} .$$

On the other hand

$$R_s = \frac{1 + i\,\frac{\sqrt{\mu_o}}{m_o}\,\theta_o}{1 - i\,\frac{\sqrt{\mu_o}}{m_o}\,\theta_o} .$$

Hence the relation $R_p = -\,R_s$ is proved.

In order to show the relation $D_p = D_s$ we consider the integral

$$-\,\frac{2\pi}{\lambda}\int_o^{\ell} m^2\Theta\,dz = \frac{2\pi}{\lambda}\int_o^{\ell}\frac{m^2}{\theta}\,dz .$$

From (51)

$$\frac{2\pi}{\lambda}\,\frac{m^2}{\theta} = \frac{\theta'}{\theta} - \frac{2\pi}{\lambda}\,\mu\,\theta .$$

Hence

$$\frac{2\pi}{\lambda}\int_o^{\ell}\frac{m^2}{\theta}\,dz = \log\frac{\theta(\ell)}{\theta(0)} - \frac{2\pi}{\lambda}\int_o^{\ell}\mu\,\theta\,dz ,$$

and

$$D_p = \frac{2\sqrt{\frac{\mu_1}{\mu_o}}\,\frac{m_o}{m_1}}{1 + \frac{m_o}{\sqrt{\mu_o}}\,\frac{1}{\theta_o}} \cdot \frac{\theta(\ell)}{\theta_o} \cdot e^{-\frac{2\pi}{\lambda}\int_o^{\ell}\mu\theta\,dz} .$$

We introduce the boundary condition $\theta(\ell) = i\,\frac{m_1}{\sqrt{\mu_1}}$ and find

$$D_p = \frac{2\,i\,\frac{m_o}{\sqrt{\mu_o}}}{\theta_o + i\,\frac{m_o}{\sqrt{\mu_o}}} \cdot e^{-\frac{2\pi}{\lambda}\int_o^{\ell}\mu\theta\,dz} ,$$

or

$$D_p = \frac{2}{1 - i\frac{\sqrt{\mu_o}}{m_o}\,\theta_o} \cdot e^{-\frac{2\pi}{\lambda}\int_o^{\ell}\mu\theta\,dz},$$

which is identical with the expression for $D_s$ if the substitution $M_o = m_o\sqrt{\mu_o}$ is made.

From these considerations it follows that in case of normal incidence only the quantities $R_s$ and $D_s$ need to be considered. We formulate our results for this case, calling

$$R = R_s \text{ and } D = D_s$$

the reflection and transmission respectively, for normal incident waves.

1) Continuous Variation:

Let $\theta(z)$ be the solution of the Riccati Equation

$$(52) \qquad \theta' = \frac{2\pi}{\lambda}(m^2 + \mu\theta^2)$$

which satisfies the boundary condition

$$(52') \qquad \theta(\ell) = i\,\frac{m_1}{\sqrt{\mu_1}}$$

Then the reflection and transmission of the layer-system for normal incident waves is given by the expressions:

$$(53) \qquad R = \frac{1 + i\frac{\sqrt{\mu_o}}{m_o}\,\theta_o}{1 - i\frac{\sqrt{\mu_o}}{m_o}\,\theta_o},$$

$$(54) \qquad D = \frac{2}{1 - i\frac{\sqrt{\mu_o}}{m_o}\,\theta_o}\; e^{-\frac{2\pi}{\lambda}\int_o^{\ell}\mu\theta\,dz}.$$

2) The Multilayer:

Let $\Theta_\nu$ be the solution of the recursion formula

(55) $$\Theta_{\nu-1} = \frac{\Theta_\nu - \frac{m_\nu}{\sqrt{\mu_\nu}} \tan (m_\nu \sqrt{\mu_\nu}\, T_\nu)}{1 + \Theta_\nu \frac{\sqrt{\mu_\nu}}{m_\nu} \tan (m_\nu \sqrt{\mu_\nu}\, T_\nu)}$$

which satisfies the boundary conditions

(55') $$\Theta_r = i \frac{m_{r+1}}{\sqrt{\mu_{r+1}}}$$

Then reflection and transmission of the multilayer for a normally incident wave are given by the expressions:

(56) $$R = \frac{1 + i \frac{\sqrt{\mu_o}}{m_o} \Theta_o}{1 - i \frac{\sqrt{\mu_o}}{m_o} \Theta_o} ,$$

(57) $$D = \frac{2}{1 - i \frac{\sqrt{\mu_o}}{m_o} \Theta_o} \prod_{\nu=1}^{r} \frac{1}{\cos m_\nu \sqrt{\mu_\nu}\, T_\nu + \frac{\sqrt{\mu_\nu}}{m_\nu} \Theta_\nu \sin m_\nu \sqrt{\mu_\nu}\, T_\nu}$$

11. Conservation of Energy.

In many applications one is mainly interested in the absolute values of R and D, i.e., in the quantities $|R|^2$ and $|D|^2$, which determine the intensities of the reflected and transmitted waves respectively. If that is the case it may not be necessary to carry out the integrations in (47) in order to find $|D_p|^2$ and $|D_s|^2$. $R_p$ and $R_s$ are obtained directly from (46), and in case the first two media, i.e., the lower boundary medium and the layer, are dielectrics, $|D_p|^2$ and $|D_s|^2$ can be related to $R_p$ and $R_s$ by formulas resulting from the general statement of conservation of radiation energy. In order to derive this statement we consider first the functions U and V which satisfy equations (11). We shall assume that

the lower boundary medium is a dielectric; hence $\sigma_0 = 0$ and $m_0^2$ is real.

The function U and its conjugate function $\overline{U}$ satisfy the equations

$$\mu\left(\frac{U'}{\mu}\right)' + \frac{\omega^2}{c^2} M^2 U = 0 ,$$

$$\mu\left(\frac{\overline{U}'}{\mu}\right)' + \frac{\omega^2}{c^2} \overline{M}^2 \overline{U} = 0 .$$

We multiply the first equation by $\overline{U}$ and the second equation by U and subtract the results. This gives us

$$\mu\left[\overline{U}\left(\frac{U'}{\mu}\right)' - U\left(\frac{\overline{U}'}{\mu}\right)'\right] + \frac{\omega^2}{c^2}\left(M^2 - \overline{M}^2\right) U \overline{U} = 0.$$

Since

$$\overline{U}\left(\frac{U'}{\mu}\right)' - U\left(\frac{\overline{U}'}{\mu}\right)' = \frac{d}{dz}\left[\frac{\overline{U}\,U' - U\,\overline{U}'}{\mu}\right] ,$$

and

$$M^2 - \overline{M}^2 = \mu\left(m^2 - \overline{m}^2\right) = -\mu\,\frac{8\pi\sigma}{\omega}\, i ,$$

it follows that

$$\frac{ic^2}{8\pi\omega}\,\frac{d}{dz}\left[\frac{\overline{U}\,U' - U\,\overline{U}'}{\mu}\right] + \sigma |U|^2 = 0 . \tag{58}$$

A similar equation can be derived for the function V. We have

$$\left(\frac{V'}{m^2}\right)' + \frac{\omega^2}{c^2}\,\frac{M^2}{m^2}\, V = 0 ,$$

$$\left(\frac{\overline{V}'}{\overline{m}^2}\right)' + \frac{\omega^2}{c^2}\,\frac{\overline{M}^2}{\overline{m}^2}\, \overline{V} = 0 .$$

We multiply the first equation by $\overline{V}$, the second by V and subtract obtaining

$$\overline{V}\left(\frac{V'}{m^2}\right)' - V\left(\frac{\overline{V}'}{\overline{m}^2}\right)' + \frac{\omega^2}{c^2}\left(\frac{M^2}{m^2} - \frac{\overline{M}^2}{\overline{m}^2}\right) V\,\overline{V} = 0 .$$

Since

$$\frac{d}{dz}\left[\frac{\overline{V}\,V'}{m^2} - \frac{V\,\overline{V}'}{\overline{m}^2}\right] = \overline{V}\left(\frac{V'}{m^2}\right)' - V\left(\frac{\overline{V}'}{\overline{m}^2}\right)' + V'\,\overline{V}'\left(\frac{1}{m^2} - \frac{1}{\overline{m}^2}\right),$$

we can write

$$\frac{d}{dz}\left[\frac{\bar{V}\,V'}{m^2}-\frac{V\,\bar{V}'}{\bar{m}^2}\right]-|V'|^2\left(\frac{1}{m^2}-\frac{1}{\bar{m}^2}\right)+\frac{\omega^2}{c^2}\left(\frac{M^2}{m^2}-\frac{\bar{M}^2}{\bar{m}^2}\right)|V|^2=0.$$

But

$$\frac{1}{m^2}-\frac{1}{\bar{m}^2}=\frac{\bar{m}^2-m^2}{m^2\,\bar{m}^2}=\frac{8\pi\sigma i}{\omega m^2\,\bar{m}^2},$$

$$\frac{M^2}{m^2}-\frac{\bar{M}^2}{\bar{m}^2}=m_o^2\mu_o\,p_o^2\left(\frac{1}{\bar{m}^2}-\frac{1}{m^2}\right)=-m_o^2\mu_o\,p_o^2\,\frac{8\pi\sigma i}{\omega m^2\,\bar{m}^2}.$$

Hence

$$\frac{d}{dz}\left[\frac{\bar{V}\,V'}{m^2}-\frac{V\,\bar{V}'}{\bar{m}^2}\right]-\frac{8\pi\sigma i}{\omega m^2\,\bar{m}^2}\left[|V'|^2+\frac{\omega^2}{c^2}\,m_o^2\mu_o\,p_o^2\,|V|^2\right]=0.$$

or finally,

$$\text{(59)}\qquad \frac{ic^2}{8\pi\omega}\,\frac{d}{dz}\left[\frac{\bar{V}\,V'}{m^2}-\frac{V\,\bar{V}'}{\bar{m}^2}\right]+\sigma\left[\frac{c^2}{\omega^2 m^2\bar{m}^2}\,|V'|^2+\frac{m_o^2\mu_o p_o^2}{m^2\bar{m}^2}\,|V|^2\right]=0.$$

The two relations (58) and (59) contain the statement of conservation of energy in our electric field. In fact they are equivalent to the well-known energy relation which can be derived directly from Maxwell's equations (1), as we shall show.* It is readily verified that equations (1) are satisfied by the conjugates $\bar{E}$ and $\bar{H}$ as well as by E and H, which are complex vectors of the form $E' + iE''$ and $H' + iH''$. Combining equations (1) after suitable scalar multiplications in order to obtain the expression on the left below, we have

$$\bar{E}\operatorname{curl}H-H\operatorname{curl}\bar{E}+E\operatorname{curl}\bar{H}-\bar{H}\operatorname{curl}E=\frac{8\pi\sigma}{c}E\,\bar{E}+\frac{1}{c}\,\frac{\partial}{\partial t}\left(\varepsilon E\bar{E}+\mu H\bar{H}\right),$$

or, on account of the vector identity, valid for complex vectors,

$$A\operatorname{curl}B-B\operatorname{curl}A=\operatorname{div}\left[B\times A\right],$$

$$-\operatorname{div}\left[(E\times\bar{H})+(\bar{E}\times H)\right]=\frac{8\pi\sigma}{c}|E|^2+\frac{1}{c}\frac{\partial}{\partial t}\left[\varepsilon|E|^2+\mu|H|^2\right].$$

---

* The derivation of equation (61) is similar to that found in texts on electromagnetic theory. cf Stratton, p 133.

We now introduce the vector

$$(60) \qquad S = \frac{c}{8\pi}\left[\left(E \times \bar{H}\right) + \left(\bar{E} \times H\right)\right],$$

which is known as Poynting's Radiation vector. We obtain the familiar statement

$$(61) \qquad \frac{\partial}{\partial t}\left[\frac{1}{8\pi}\left(\varepsilon|E|^2 + \mu|H|^2\right)\right] + \sigma|E|^2 + \operatorname{div} S = 0.$$

The change of the electromagnetic energy $\frac{1}{8\pi}\left(\varepsilon|E|^2 + \mu|H|^2\right)$, the produced heat $\sigma|E|^2$, and the radiation div S, add up to zero.

The statement (61) simplifies for the stationary solution

$$E = u e^{i\omega t},$$
$$H = v e^{i\omega t}.$$

The electromagnetic energy

$$\frac{1}{8\pi}\left[\varepsilon|E|^2 + \mu|H|^2\right] = \frac{1}{8\pi}\left[\varepsilon|u|^2 + \mu|v|^2\right]$$

becomes independent of t. Hence we have the relation

$$(62) \qquad \operatorname{div} S + \sigma|u|^2 = 0,$$

in which

$$(63) \qquad S = \frac{c}{8\pi}\left[\left(u \times \bar{v}\right) + \left(\bar{u} \times v\right)\right].$$

We have found before that the two vectors u and v could be expressed by the two scalar functions U and V. On account of (8) and (6):

$$u_1 = i\frac{c}{m^2\omega}\, e^{-i\frac{\omega}{c} m_o\sqrt{\mu_o}\, p_o x}\, V' \quad ; \quad v_1 = -i\frac{c}{\mu\omega}\, e^{-i\frac{\omega}{c} m_o\sqrt{\mu_o}\, p_o x}\, U';$$

$$u_2 = e^{-i\frac{\omega}{c} m_o\sqrt{\mu_o}\, p_o x}\, U \quad ; \quad v_2 = e^{-i\frac{\omega}{c} m_o\sqrt{\mu_o}\, p_o x}\, V;$$

$$u_3 = -\frac{m_o\sqrt{\mu_o}\, p_o}{m^2}\, e^{-i\frac{\omega}{c} m_o\sqrt{\mu_o}\, p_o x}\, V \; ; \; v_3 = \frac{m_o\sqrt{\mu_o}\, p_o}{\mu}\, e^{-i\frac{\omega}{c} m_o\sqrt{\mu_o}\, p_o x}\, U.$$

In order to construct the Poynting vector S we have to form the two vector products that follow. (The factors $e^{-i\frac{\omega}{c} m_o\sqrt{\mu_o}\, p_o x}$ and its conjugate are not included for they will be multiplied by each other in each component of the cross products and thus cancel out).

$$(u \times \bar{v}) = \left\| \begin{matrix} \frac{ic}{m^2\omega} V' & , & U & , & -\frac{m_o\sqrt{\mu_o}\, p_o}{m^2} V \\ \frac{ic}{\mu\omega} \bar{U}' & , & \bar{V} & , & \frac{m_o\sqrt{\mu_o}\, p_o}{\mu} \bar{U} \end{matrix} \right\|$$

$$(\bar{u} \times v) = \left\| \begin{matrix} -\frac{ic}{\bar{m}^2\omega} \bar{V}' & , & \bar{U} & , & -\frac{m_o\sqrt{\mu_o}\, p_o}{\bar{m}^2} \bar{V} \\ -\frac{ic}{\mu\omega} U' & , & V & , & \frac{m_o\sqrt{\mu_o}\, p_o}{\mu} U \end{matrix} \right\|$$

Obviously all components $S_1, S_2, S_3$ of the Poynting vector depend on z only. Hence

$$\operatorname{div} S = \frac{\partial S_1}{\partial x} + \frac{\partial S_2}{\partial y} + \frac{\partial S_3}{\partial z} = \frac{\partial S_3}{\partial z} .$$

Thus, explicitly

$$\text{(64)} \qquad \operatorname{div} S = \frac{ic^2}{8\pi\omega} \frac{d}{dz} \left[ \frac{\bar{U}U' - U\bar{U}'}{\mu} + \frac{\bar{V}V'}{m^2} - \frac{V\bar{V}'}{\bar{m}^2} \right]$$

On the other hand

$$\text{(65)} \qquad |u|^2 = |U|^2 + \frac{c^2}{\omega^2 m^2 \bar{m}^2} |V'|^2 + \frac{m_o^2 \mu_o p_o^2}{m^2 \bar{m}^2} |V|^2 .$$

By introducing (64) and (65) in (62) the sum of the relations (58) and (59) is obtained, justifying our interpretation of (58) and (59) as equivalent to the energy relation (62).

Equations (58) and (59) become relations for the functions a,b; A,B if U,U'; V, V' are replaced by using equations (30).

The result is

$$\frac{c}{8\pi}\frac{d}{dz}\left[\left(\frac{M+\bar{M}}{\mu}\right)(a\bar{a}-b\bar{b})+\left(\frac{M-\bar{M}}{\mu}\right)(a\bar{b}-b\bar{a})\right]=\sigma|a+b|^2 ,$$

(66)

$$\frac{c}{8\pi}\frac{d}{dz}\left[\left(\frac{M}{m^2}+\frac{\bar{M}}{\bar{m}^2}\right)(A\bar{A}-B\bar{B})+\left(\frac{M}{m^2}-\frac{M}{\bar{m}^2}\right)(A\bar{B}-B\bar{A})\right]=\frac{\sigma}{m^2\bar{m}^2}\left[|M|^2|A-B|^2+m_o^2\mu_o p_o|A+B|^2\right].$$

We have assumed that the first medium $\varepsilon_o$, $\sigma_o$, $\mu_o$ is a dielectric medium, i.e. $\sigma_o = 0$, $M_o$ real. We integrate (66) from 0 to $\ell$ and remember $M_o = \bar{M}_o$ and $a(\ell) = A(\ell) = 0$.

We obtain

$$\frac{c}{8\pi}\left[-\frac{M_1+\bar{M}_1}{\mu_1}|b(\ell)|^2-\frac{2M_o}{\mu_o}(|a_o|^2-|b_o|^2)\right]=\int_o^\ell\sigma|a+b|^2\,dz ,$$

$$\frac{c}{8\pi}\left[-\left(\frac{M_1}{m_1^2}+\frac{\bar{M}_1}{\bar{m}_1^2}\right)|B(\ell)|^2-\frac{2M_o}{m_o^2}\left(|A_o|^2-|B_o|^2\right)\right]=\int_o^\ell\frac{\sigma\,dz}{m^2\bar{m}^2}\left[|M|^2|A-B|^2+m_o^2\mu_o p_o^2|A+B|^2\right],$$

or

$$\frac{M_o}{\mu_o}|b_o|^2=\frac{M_o}{\mu_o}|a_o|^2+\frac{M_1+\bar{M}_1}{2\mu_1}|b(\ell)|^2+\frac{4\pi}{c}\int_o^\ell\sigma|a+b|^2\,dz$$

(67)

$$\frac{M_o}{m_o^2}|B_o|^2=\frac{M_o}{m_o^2}|A_o|^2+\frac{1}{2}\left(\frac{M_1}{m_1^2}+\frac{\bar{M}_1}{\bar{m}_1^2}\right)|B(\ell)|^2+\frac{4\pi}{c}\int_o^\ell\frac{\sigma\,dz}{m^2\bar{m}^2}\left[|M|^2|A-B|^2+m_o^2\mu_o p_o^2|A+B|^2\right]$$

or finally

$$\frac{M_o}{\mu_o}=\frac{M_o}{\mu_o}\left|\frac{a_o}{b_o}\right|^2+\frac{M_1+\bar{M}_1}{2\mu_1}\left|\frac{b(\ell)}{b_o}\right|^2+\frac{4\pi}{c}\int_o^\ell\sigma\left|\frac{a}{b_o}+\frac{b}{b_o}\right|^2 dz ,$$

$$\frac{M_o}{m_o^2} = \frac{M_o}{m_o^2}\left|\frac{A_o}{B_o}\right|^2 + \frac{1}{2}\left(\frac{M_1}{m_1^2} + \frac{\bar{M}_1}{\bar{m}_1^2}\right)\left|\frac{B(\ell)}{B_o}\right|^2 + \frac{4\pi}{c}\int_o^\ell \frac{\sigma dz}{m^2\bar{m}^2}\left[|M|^2\left|\frac{A}{B_o} - \frac{B}{B_o}\right|^2 + m_o^2\mu_o p_o^2\left|\frac{A}{B_o} + \frac{B}{B_o}\right|^2\right].$$

We introduce the following substitutions on account of (42) and (44), (45):

$$\frac{a_o}{b_o} = R_s \quad , \quad \frac{b(\ell)}{b_o} = D_s \quad ,$$

$$\frac{A_o}{B_o} = R_p \quad , \quad \frac{B(\ell)}{B_o} = D_p\sqrt{\frac{\mu_o}{\mu_1}}\ \frac{m_1}{m_o} \ .$$

Hence

$$\frac{M_o}{\mu_o} = \frac{M_o}{\mu_o}\left|R_s\right|^2 + \frac{M_1+\bar{M}_1}{2\mu_1}\left|D_s\right|^2 + \frac{4\pi}{c}\int_o^\ell \left|\frac{a+b}{b_o}\right|^2 dz \ ,$$

$$\frac{M_o}{m_o^2} = \frac{M_o}{m_o^2}\left|R_p\right|^2 + \frac{1}{2}\left(\frac{M_1}{m_1^2} + \frac{\bar{M}_1}{\bar{m}_1^2}\right)\frac{|m_1|^2}{m_o^2}\frac{\mu_o}{\mu_1}\left|D_p\right|^2$$

$$+ \frac{4\pi}{c}\int_o^\ell \frac{\sigma dz}{m^2\bar{m}^2}\left[|M|^2\left|\frac{A}{B_o} - \frac{B}{B_o}\right|^2 + m_o^2\mu_o p_o^2\left|\frac{A}{B_o} + \frac{B}{B_o}\right|^2\right].$$

or finally:

$$\frac{M_o}{\mu_o}\left(1 - |R_s|^2\right) = \frac{M_1+\bar{M}_1}{2\mu_1}\left|D_s\right|^2 + \frac{4\pi}{c}\int_o^\ell \sigma\left|\frac{a+b}{b_o}\right|^2 dz,$$

(68)

$$\frac{M_o}{\mu_o}\left(1 - |R_p|^2\right) = \frac{|m_1|^2}{2\mu_1}\left(\frac{M_1}{m_1^2} + \frac{\bar{M}_1}{\bar{m}_1^2}\right)\left|D_p\right|^2$$

$$+ \frac{4\pi m_o^2}{c\mu_o}\int_o^\ell \frac{\sigma dz}{m^2\bar{m}^2}\left[|M|^2\left|\frac{A-B}{B_o}\right|^2 + m_o^2\mu_o p_o^2\left|\frac{A+B}{B_o}\right|^2\right].$$

These equations are especially valuable in case that the layer $0 < z < \ell$ consists entirely of dielectric substances. In this case $\sigma = 0$, and hence

$$\frac{M_o}{\mu_o}\left(1 - |R_s|^2\right) = \frac{M_1 + \bar{M}_1}{2\mu_1}\,|D_s|^2 \quad ,$$

(69)

$$\frac{M_o}{\mu_o}\left(1 - |R_p|^2\right) = \frac{|m_1|^2}{2\mu_1}\left(\frac{M_1}{m_1^2} + \frac{\bar{M}_1}{\bar{m}_1^2}\right)|D_p|^2 \quad .$$

If the last medium, $z > \ell$, is also dielectric, we have

$$\frac{M_o}{\mu_o}\left(1 - |R_s|^2\right) = \frac{M_1 + \bar{M}_1}{2\mu_1}\,|D_s|^2 \quad ,$$

(70)

$$\frac{M_o}{\mu_o}\left(1 - |R_p|^2\right) = \frac{M_1 + \bar{M}_1}{2\mu_1}\,|D_p|^2 \quad .$$

Since $M_1 = \sqrt{m_1^2\mu_1 - m_o^2\mu_o\,p_o^2}$ we conclude that $M_1$ is either real or pure imaginary, depending on the sign of $m_1^2\mu_1 - m_o^2\mu_o\,p_o^2$.

In the case, $m_o^2\mu_o\,p_o^2 < m_1^2\mu_1$, we thus find

$$\frac{M_o}{\mu_o}\left(1 - |R_s|^2\right) = \frac{M_1}{\mu_1}\,|D_s|^2 \quad ,$$

(71)

$$\frac{M_o}{\mu_o}\left(1 - |R_p|^2\right) = \frac{M_1}{\mu_1}\,|D_p|^2 \quad .$$

In the case, $m_o^2\mu_o\,p_o^2 > m_1^2\mu_1$ we find

(72) $$|R_s|^2 = |R_p|^2 = 1 \ ,$$

which is the case of <u>Total Reflection</u>.

## 12. Two Media Separated by One Surface. Fresnel's Formulae.

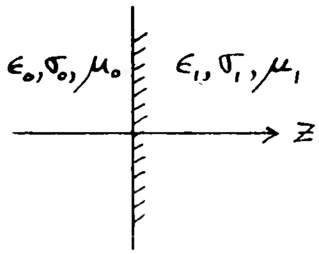



Fig. 9

We apply our formulae to the simplest case of two media separated by a plane surface. Letting r = 0 in the expressions for $\theta_r$ and $\Theta_r$ in (19) we obtain for our special case

$$\theta_0 = i\,\frac{M_1}{\mu_1}\,, \qquad \Theta_0 = i\,\frac{M_1}{m_1^2}\,.$$

Hence, using (48) and (49),

$$R_p = \frac{1 - \frac{M_1}{M_0}\frac{m_0^2}{m_1^2}}{1 + \frac{M_1}{M_0}\frac{m_0^2}{m_1^2}}\,, \qquad R_s = \frac{1 - \frac{M_1}{M_0}\frac{\mu_0}{\mu_1}}{1 + \frac{M_1}{M_0}\frac{\mu_0}{\mu_1}}\,,$$

(73)

$$D_p = \frac{2\sqrt{\frac{\mu_1}{\mu_0}}\,\frac{m_0}{m_1}}{1 + \frac{M_1}{M_0}\frac{m_0^2}{m_1^2}}\,, \qquad D_s = \frac{2}{1 + \frac{M_1}{M_0}\frac{\mu_0}{\mu_1}}\,.$$

These formulas are valid for any two media. They agree with Fresnel's Formulae in the special case of two dielectric media. We introduce $M_1 = m_1 \cos\varphi_1$, $M_0 = m_0 \cos\varphi_0$; $\mu_0 = \mu_1 = 1$ and find

$$R_p = \frac{1 - \frac{m_0}{m_1}\,\frac{\cos\varphi_1}{\cos\varphi_0}}{1 + \frac{m_0}{m_1}\cdot\frac{\cos\varphi_1}{\cos\varphi_0}}\,, \qquad R_s = \frac{1 - \frac{m_1}{m_0}\,\frac{\cos\varphi_1}{\cos\varphi_0}}{1 + \frac{m_1}{m_0}\,\frac{\cos\varphi_1}{\cos\varphi_0}}\,,$$

$$D_p = \frac{2\,\frac{m_0}{m_1}}{1 + \frac{m_0}{m_1}\,\frac{\cos\varphi_1}{\cos\varphi_0}}\,, \qquad D_s = \frac{2}{1 + \frac{m_1}{m_0}\,\frac{\cos\varphi_1}{\cos\varphi_0}}\,.$$

By introducing $\frac{m_o}{m_1} = \frac{\sin\varphi_1}{\sin\varphi_o}$ in these formulas Fresnel's relations are obtained in the customary form:*

$$R_p = -\frac{\tan(\varphi_1 - \varphi_o)}{\tan(\varphi_1 + \varphi_o)}\,, \qquad R_s = \frac{\sin(\varphi_1 - \varphi_o)}{\sin(\varphi_1 + \varphi_o)}\,,$$

(74)

$$D_p = \frac{2\sin\varphi_1 \cos\varphi_o}{\sin(\varphi_o + \varphi_1)\cos(\varphi_o - \varphi_1)}\,, \qquad D_s = \frac{2\sin\varphi_1 \cos\varphi_o}{\sin(\varphi_o + \varphi_1)}\,.$$

In case of normal incidence, $M = m\sqrt{\mu}$ and formulae (73) reduce to

$$R = -R_p = R_s = -\frac{1 - \frac{m_o}{m_1}\sqrt{\frac{\mu_1}{\mu_o}}}{1 + \frac{m_o}{m_1}\sqrt{\frac{\mu_1}{\mu_o}}}\,;$$

(74')

$$D = D_p = D_s = \frac{2}{1 + \frac{m_1}{m_o}\sqrt{\frac{\mu_o}{\mu_1}}}\,.$$

13. Three Media Separated by Two Plane Surfaces.

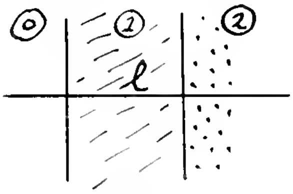



Fig. 10

We consider next the case that the medium ①(Fig. 10) occupies the whole layer $0 < z < \ell$. We apply equations (19), (48) and (49) .

First from (19) $\theta_1 = i\,\frac{M_2}{\mu_2}\,, \qquad \Theta_1 = i\,\frac{M_2}{m_2^2}\,,$

$$\theta_o = \frac{\theta_1 - \frac{M_1}{\mu_1}\tan M_1 T_1}{1 + \theta_1 \frac{\mu_1}{M_1}\tan M_1 T_1}\,, \qquad \Theta_o = \frac{\Theta_1 - \frac{M_1}{m_1^2}\tan M_1 T_1}{1 + \Theta_1 \frac{m_1^2}{M_1}\tan M_1 T_1}\,.$$

*cf Stratton, p 496

Hence, combining these formulae,

$$\theta_o = \frac{i\,\dfrac{M_2}{\mu_2} - \dfrac{M_1}{\mu_1}\tan M_1T_1}{1 + i\,\dfrac{M_2}{\mu_2}\dfrac{\mu_1}{M_1}\tan M_1T_1}\,, \qquad \Theta_o = \frac{i\,\dfrac{M_2}{m_2^2} - \dfrac{M_1}{m_1^2}\tan M_1T_1}{1 + i\,\dfrac{M_2}{m_2^2}\dfrac{m_1^2}{M_1}\tan M_1T_1}\,.$$

We introduce these quantities $\theta_o, \Theta_o, \theta_1, \Theta_1$, in the equations (48) and (49) and obtain the reflection and transmission of the layer as functions of the thickness, $T_1 = 2\pi\frac{\ell}{\lambda}$, of the layer. The results are

(75)

$$R_p = \frac{\left(1 - \dfrac{m_o^2}{M_o}\dfrac{M_2}{m_2^2}\right)\cos M_1T_1 + i\left(\dfrac{M_2}{m_2^2}\dfrac{m_1^2}{M_1} - \dfrac{m_o^2}{M_o}\dfrac{M_1}{m_1^2}\right)\sin M_1T_1}{\left(1 + \dfrac{m_o^2}{M_o}\dfrac{M_2}{m_2^2}\right)\cos M_1T_1 + i\left(\dfrac{M_2}{m_2^2}\dfrac{m_1^2}{M_1} + \dfrac{m_o^2}{M_o}\dfrac{M_1}{m_1^2}\right)\sin M_1T_1}\,,$$

$$D_p = \frac{2\sqrt{\dfrac{\mu_2}{\mu_o}}\;\dfrac{m_o}{m_2}}{\left(1 + \dfrac{m_o^2}{M_o}\dfrac{M_2}{m_2^2}\right)\cos M_1T_1 + i\left(\dfrac{M_2}{m_2^2}\dfrac{m_1^2}{M_1} + \dfrac{m_o^2}{M_o}\dfrac{M_1}{m_1^2}\right)\sin M_1T_1}\,,$$

(76)

$$R_s = \frac{\left(1 - \dfrac{\mu_o}{M_o}\dfrac{M_2}{\mu_2}\right)\cos M_1T_1 + i\left(\dfrac{M_2}{\mu_2}\dfrac{\mu_1}{M_1} - \dfrac{\mu_o}{M_o}\dfrac{M_1}{\mu_1}\right)\sin M_1T_1}{\left(1 + \dfrac{\mu_o}{M_o}\dfrac{M_2}{\mu_2}\right)\cos M_1T_1 + i\left(\dfrac{M_2}{\mu_2}\dfrac{\mu_1}{M_1} + \dfrac{\mu_o}{M_o}\dfrac{M_1}{\mu_1}\right)\sin M_1T_1}\,,$$

$$D_s = \frac{2}{\left(1 + \dfrac{\mu_o}{M_o}\dfrac{M_2}{\mu_2}\right)\cos M_1T_1 + i\left(\dfrac{M_2}{\mu_2}\dfrac{\mu_1}{M_1} + \dfrac{\mu_o}{M_o}\dfrac{M_1}{\mu_1}\right)\sin M_1T_1}\,.$$

In the special case of normal incidence,

$$R=-R_p=R_s=\frac{\left(1-\frac{m_2}{m_0}\sqrt{\frac{\mu_0}{\mu_2}}\right)\cos m_1\sqrt{\mu_1}\,T_1+i\left(\frac{m_2}{m_1}\sqrt{\frac{\mu_1}{\mu_2}}-\frac{m_1}{m_0}\sqrt{\frac{\mu_0}{\mu_1}}\right)\sin m_1\sqrt{\mu_1}\,T_1}{\left(1+\frac{m_2}{m_0}\sqrt{\frac{\mu_0}{\mu_2}}\right)\cos m_1\sqrt{\mu_1}\,T_1+i\left(\frac{m_2}{m_1}\sqrt{\frac{\mu_1}{\mu_2}}+\frac{m_1}{m_0}\sqrt{\frac{\mu_0}{\mu_1}}\right)\sin m_1\sqrt{\mu_1}\,T_1}\,.$$

(77)

$$D=D_p=D_s=\frac{2}{\left(1+\frac{m_2}{m_0}\sqrt{\frac{\mu_0}{\mu_2}}\right)\cos m_1\sqrt{\mu_1}\,T_1+i\left(\frac{m_2}{m_1}\sqrt{\frac{\mu_1}{\mu_2}}+\frac{m_1}{m_0}\sqrt{\frac{\mu_0}{\mu_1}}\right)\sin m_1\sqrt{\mu_1}\,T_1}\,.$$

## 14. Conclusion.

The theory presented here is complete in itself. All the desirable information on the behavior of a plane wave incident upon either a series of layers, each with constant $\varepsilon, \mu, \sigma$, or incident upon a layer with continuously varying $\varepsilon, \mu, \sigma$, can be computed from the formulas derived. Particular attention should be called to the fact that the theory contains as special cases many known results on the behavior of a plane wave incident on a plane surface separating two media.

The problem of determining an electromagnetic field under any given conditions is the problem of determining six quantities, the three components of the electric field and the three components of the magnetic field. Such problems are often attacked by means of scalar and vector potentials from which the electromagnetic field can be derived, or by working with the Hertz vector from which, also, the electromagnetic field can be derived. This paper reduces the problem of finding the field to the solution of two ordinary differential equations of second order, which are then transformed into Riccati equations. This reduction of the solution of Maxwell's equations to ordinary differential equations is not in itself new. It is the special simplicity of the equations here used which merits some attention.

The relationship of this paper to the attack on the problem of radio wave propagation in non-homogeneous atmospheres should now be noted. It has already been remarked that the approach contemplated is that of

representing an arbitrary source as an integral of plane waves. (For some indication of earlier work along such lines see the discussion in Stratton p. 577 ff.). With the behavior of plane waves in non-homogeneous media known, a major step in this plan is accomplished. The theory will ultimately lead to a complex integral and it is likely that much difficulty may be encountered in the evaluation of this integral, even approximately. Nevertheless, this approach seems to warrant consideration as compared with previous efforts made in this field.

It must be remarked further that the theory of this paper, as well as the extension just mentioned, presupposes a flat earth. To take care of the earth's curvature it will probably be necessary to use some modification of the true index of refraction, such as has been employed in other studies.

## Date Due

| | | | |
|---|---|---|---|
| FEB 15 '61 | | | |
| | | | |
| | | | |
| | | | |
| | | | |
| | | | |
| | | | |
| JUL 26 '66 | | | |
| | | | |
| | | | |
| | | | |
| | | | |
| | | | |
| | | | |
| | | | |
| | | | |
| | | | |

PRINTED IN U. S. A.

N. Y. U. Institute of
Mathematical Sciences
25 Waverly Place
New York 3, N. Y.